ALGEBRA

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FOR SCHOOL CERTIFICATE AND MATRICULATION

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PREFACE

In no subject can a text-book replace the really capable and imaginative teacher. If unlimited time were at the disposal of such a teacher of Algebra, all that would be required would be a vast number of exercises for the pupil. In this ideal case no text at all would be necessary.

In practice, however, the teacher has not unlimited time; some pupils are absent for certain lessons; other pupils, either due to temporary fatigue or to general slowness, miss the essentials of various lessons. For these reasons a text becomes necessary, and that text should model itself on the teacher at his best. Many text-books have been written mainly on the formal work in Algebra, with the spirit of the ideal classroom teacher missing.

One aim which the authors have set themselves has been to lead up to all sections of formal work by discussion, intended to arouse the interest of the reader, to show the need for the manipulative work which follows, and to pave the way for an understanding of that work. They believe that in this way the text has been made readable; the pupils' interest being aroused from section to section, and the work in each section being shown to have an intelligible application. In furtherance of this aim, much thought has been given to the choice of words, which, particularly in Part I, have been made as simple as possible.

While every endeavour has been made to keep the book readable throughout, great care has been taken to ensure that the formal work and the necessary proofs are as complete and rigid as is possible in a book of this standard. Fundamental principles have been explained in a way the pupil can grasp, and methods of working have been fully justified.

The examples worked out in the text cover a wide range, are very varied in type, and include many problems of a practical

nature, such as are met with in physics, mechanics, etc. They have been chosen to illustrate important points, which are frequently emphasised by additional notes. Various short methods and devices for easy manipulation, some of them novel to Algebra text-books, have been incorporated.

The exercises have been placed at the end of the Chapters to avoid discontinuity in the reading, the appropriate place where they should be worked being indicated in the text. These exercises have been framed to meet a widely expressed need of teachers of Algebra, being very carefully graded, exceptionally comprehensive, and more numerous than is usual.

Throughout the book the recommendations of the Mathematical Association and of the Board of Education have been considered.

The syllabuses for the School Leaving Certificates and the Matriculation Examinations of the various Examining Boards throughout Great Britain have been fully covered.

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CHAPTER XI

Simultaneous Equations

52. We have seen in Chapter IX that equations which involve only one unknown letter may be solved by simple algebraic methods. In this chapter we shall consider the solution of linear equations containing more than one unknown quantity.

2 p.m. L⟶55 m.p.h. 60 m.p.h.⟵E 3 p.m.
London——————400 miles——————Edinburgh

FIG. 6

The diagram represents a train L leaving London at 2 p.m., travelling at 55 m.p.h. to Edinburgh, 400 miles away; also a train E leaving Edinburgh for London at 3 p.m. at 60 m.p.h. In the first place, let us consider train L only. After travelling for y hours it will be at a point A, whose distance from London is $55y$ miles. Denoting this distance by x miles, we have

$$x=55y \quad . \quad . \quad . \quad . \quad . \quad (1)$$

⟵——— $x=55y$ ———⟶ A ——————————

London Edinburgh

FIG. 7

It is clear that equation (1) is satisfied by an unlimited number of pairs of values of x and y.

Thus, if $y=1$, $x=55$; if $y=2$, $x=110$; if $y=3$, $x=165$; if $y=4$, $x=220$; if $y=5$, $x=275$; etc.

For any chosen value of y, there is a definite point A reached, and a definite value of x which may be obtained from equation (1).

This illustrates the fact that when we are given a single equation in two unknowns, we can find as many pairs of values of the unknowns satisfying it as we please. Such an equation is called **indeterminate**.

Let us now consider train E, which has been travelling for one hour less than train L, that is for $(y-1)$ hours. It will have reached a point B, whose distance from Edinburgh is $60(y-1)$ miles, and therefore whose distance in miles from London is $400-60(y-1)$. Denoting this distance by x miles, we have

$$x=400-60(y-1) \quad . \quad . \quad . \quad (2)$$

$\longleftarrow x=400-60(y-1) \longrightarrow B$ ————————————

London Edinburgh

FIG. 8

Equation (2) is also indeterminate, and is satisfied by an unlimited number of pairs of values of x and y.

Thus, if $y=1$, $x=400$; if $y=2$, $x=340$; if $y=3$, $x=280$; if $y=4$, $x=220$; if $y=5$, $x=160$; etc.

It is clear that for any value of y taken at random, the values of x for L and E will not be the same; in other words, the points A and B will not coincide. When L and E meet, however, A and B *do* coincide. Hence for the value of y when they meet, and for **this value of y only,** the values of x for L and E are equal. Thus we are led to the conclusion that **there is only one pair of values of x and y which satisfy both equations (1) and (2) at the same time, that is simultaneously.**

If the student refers back to the pairs of values of x and y given for equations (1) and (2), he will see that the values $y=4$, $x=220$, satisfy both equations simultaneously. This shows that the trains meet 4 hours after 2 p.m., that is at 6 p.m., at a distance of 220 miles from London.

The values, $y=4$, $x=220$, have been obtained by chance. To find with certainty when and where the trains meet, we must make A and B coincide.

$x=55y$ A

London $x=400-60(y-1)$ B Edinburgh

FIG. 9

Since the values of x are equal for the same values of y, we may equate them,

$$\therefore \quad 55y=400-60(y-1)$$
$$=400-60y+60,$$
$$\therefore \quad 115y=460, \qquad \therefore \quad y=4.$$

But $x=55y$. Hence when $y=4$, $x=55\times 4=220$.

Alternatively, $x=400-60(y-1)$.

Hence when $y=4$, $x=400-60(4-1)=220$.

It is clear that, once the correct value of y has been found, the value obtained for x must be the same **no matter which be the equation in which we substitute.**

To sum up, although there are an unlimited number of pairs of values of x and y which satisfy equation (1), and also an unlimited number which satisfy equation (2), there is only one pair which satisfy the **simultaneous equations** :

$$x=55y$$

and $$x=400-60(y-1),$$

namely : $$x=220, \quad y=4.$$

When, as in the above example, two (or more) equations are considered together, they are called simultaneous equations, and the values of the unknowns which **satisfy each of the equations** is called their solution.

Methods of Solving Simultaneous Equations

Method I

53. Example 1. Find two numbers, such that three times the first added to twice the second is 12, while four times the first number decreased by the second is 5.

Denote the two numbers by x and y respectively.

Then $$3x+2y=12 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

and $$4x-y=5 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Equation (1) may be written

$$2y=12-3x, \qquad \therefore \quad y=\tfrac{1}{2}(12-3x).$$

Equation (2) may be written $y=4x-5$.

Since y is to have the same value in each equation for the same value of x,

$$\therefore \quad \tfrac{1}{2}(12-3x)=4x-5,$$

$$\therefore \quad 12-3x=8x-10,$$

$$\therefore \quad 11x=22, \qquad \therefore \quad x=2.$$

If the value $x=2$ be substituted in either of the given equations, we find that $y=3$. Thus the solution of the pair of simultaneous equations is $x=2$, $y=3$.

Thus the two numbers are 2 and 3 respectively.

The method of solution we have employed is to reduce the two original equations in x and y to a single equation in x only. The process of removing one of the unknowns is called **elimination.** In this example we carried out **elimination by equating equivalent values of one unknown.**

54. It should be noted that in general there is no solution to two equations in one unknown.

Thus the only solution of the equation $3x+4=19$ is $x=5$, and the only solution of the equation $2x-7=1$ is $x=4$. Hence there is no value of x satisfying this pair of equations. Such a set of equations having no solution is said to be **inconsistent.**

Method II

55. Example 2. Solve

$$3x=1-7y \quad . \quad . \quad . \quad (1)$$
$$2x+3y=4 \quad . \quad . \quad . \quad . \quad (2)$$

From equation (1), $x=\frac{1}{3}(1-7y)$.

Substituting this value of x in equation (2),

$$2\times\tfrac{1}{3}(1-7y)+3y=4,$$
$$\therefore \quad 2(1-7y)+9y=12,$$
$$\text{i.e.} \quad 2-14y+9y=12,$$
$$\therefore \quad 5y=-10, \qquad \therefore \quad y=-2.$$

Substituting this value of y in (1) or (2), we obtain $x=5$.

Thus the solution is $x=5$, $y=-2$.

Verification.

Equation (1). L.H.S.$=3\times5=15$. R.H.S.$=1-7\times(-2)=15$.

Equation (2). L.H.S.$=2\times5+3\times(-2)=4$. R.H.S.$=4$.

Note.—The values obtained in the solution of simultaneous equations should always be verified for *each* of the equations given.

This method of solution is known as **Elimination by Substitution.**

Method III

56. Example 3. Solve $5x+4y=1$ . . . . (1)

$6x+5y=2$ . . . . (2)

In order to eliminate x we shall multiply equation (1) throughout by 6, and equation (2) throughout by 5, thus **making the coefficients of x the same in each equation,** namely 30.

(1)×6 $30x+24y=6,$

(2)×5 $30x+25y=10.$

Subtracting, $-y=-4,$ $\therefore\ y=4.$

Substituting this value of y in (1) or (2), we obtain $x=-3$.

Thus the solution is $x=-3,\ y=4.$

This method of solution is known as **Elimination by Equalising the Coefficients.**

Example 4. Solve $12x-22=19y$ . . . . (1)

$6+17y=8x$ . . . . (2)

The first step is to rewrite each equation with the unknowns on the L.H.S. and the constants on the R.H.S., thus:

$12x-19y=22$ . . . . (3)

$8x-17y=6$ . . . . (4)

Note.—To equalise the coefficients 12 and 8, it is necessary only to multiply the first by 2 and the second by 3, so as to make each coefficient 24, *the L.C.M. of the original coefficients.*

(3)×2 $24x-38y=44,$

(4)×3 $24x-51y=18.$

Subtracting, $13y=26,$

$\therefore\ y=2.$

Substituting this value of y in either equation we obtain $x=5$. Thus the solution is $x=5,\ y=2.$

The questions in Exercise 11a, page 132, should now be attempted.

Harder Simultaneous Equations

57. Example 5. Solve
$$\frac{12}{x}-\frac{5}{y}=25 \quad . \quad . \quad . \quad . \quad (1)$$
$$\frac{8}{x}+\frac{10}{y}=-2 \quad . \quad . \quad . \quad (2)$$

Regarding the unknowns as $\frac{1}{x}$ and $\frac{1}{y}$, we can eliminate the first term by multiplying equation (1) by 2 and equation (2) by 3, as in the note to Example 4. It is obviously simpler, however, to multiply the first equation by 2, and thus eliminate the second term.

(1)×2 $\qquad \frac{24}{x}-\frac{10}{y}=50,$

(2) $\qquad \frac{8}{x}+\frac{10}{y}=-2,$

adding, $\qquad \frac{32}{x}=48,$

$\therefore \quad \frac{1}{x}=\frac{48}{32} \qquad \therefore \quad x=\frac{32}{48}=\frac{2}{3}.$

Substituting in either equation we obtain $y=-\frac{5}{7}$.

Thus the solution is $x=\frac{2}{3}, \quad y=-\frac{5}{7}$.

Example 6. Solve $\quad 3x+4y-3=2x+y+8=x+5y-16.$

This set is equivalent to two separate equations. For we may write the first expression equal to the second, and the first expression equal to the third; in which case the second expression must obviously be equal to the third.

Thus, $\qquad 3x+4y-3=2x+y+8 \quad . \quad . \quad . \quad (1)$

$\qquad 3x+4y-3=x+5y-16 \quad . \quad . \quad . \quad (2)$

Collecting the unknowns on the L.H.S. and the constants on the R.H.S.,

(1) becomes $\qquad x+3y=11 \quad . \quad . \quad . \quad . \quad . \quad (3)$

(2) becomes $\qquad 2x-y=-13 \quad . \quad . \quad . \quad . \quad (4)$

(3)×2 $\qquad 2x+6y=22 \quad . \quad . \quad . \quad . \quad . \quad (5)$

Subtracting (5) from (4),

$$-7y=-35, \qquad \therefore \quad y=5.$$

Substituting in (3), $x=-4$.

Thus the solution is $x=-4, \quad y=5.$

The pupil should check this solution by substituting these values in the three original expressions.

Example 7. Solve

$$\tfrac{1}{5}(3x+17)+\tfrac{1}{4}(5y+12)=3 \quad . \quad . \quad . \quad (1)$$

$$\tfrac{1}{7}(4x-5y-7)-\tfrac{4}{9}(3x+2y+14\tfrac{1}{2})=-3 \quad . \quad . \quad . \quad (2)$$

The first step is to rewrite each equation, cleared of fractions, with the unknowns on the L.H.S. and the constants on the R.H.S.

$(1)\times 20$ $\quad 4(3x+17)+5(5y+12)=60,$

i.e. $12x+68+25y+60=60,$

i.e. $12x+25y=-68 \quad . \quad (3)$

$(2)\times 63$ $\quad 9(4x-5y-7)-28(3x+2y+14\tfrac{1}{2})=-189,$

i.e. $36x-45y-63-84x-56y-406=-189,$

i.e. $-48x-101y=280,$

$\therefore \quad 48x+101y=-280, \quad . \quad (4)$

$(3)\times 4$ $\quad 48x+100y=-272.$

Subtracting, $y=-8.$

Substituting in (3), $x=11.$

Thus the solution is $x=11, \quad y=-8.$

The pupil should check this solution by substituting these values in equations (1) and (2).

The questions in Exercise 11b, page 133, should now be attempted.

Simultaneous Equations with more than two Unknowns

58. In the examples considered so far, we have always had two unknowns and two equations. We may be given more than two unknowns and also more than two equations.

It was shown in § 55 that an equation may be used to make a substitution for one unknown in terms of the remaining unknown (or unknowns). If the substitution is made in all the remaining equations—in other words, if the first equation is used to eliminate

one letter from all the remaining equations, we shall be left with one unknown fewer and one equation fewer than at first. These resulting equations can now be dealt with in a similar way, and a second unknown and equation eliminated.

This process can be continued, giving rise to one of the three following cases:

1. If initially there were **as many equations as unknowns,** the final stage of the process will result in one equation in one unknown. The value of this unknown can then be determined; and by substitution the values of the other unknowns can be found in turn. Thus there will be one value only for each of the unknowns. This result is called a **unique solution.**
2. If initially there were **fewer equations than unknowns,** the final stage of the process will result in one equation in more than one unknown. This, as we have already pointed out in § 52, is **indeterminate.**
3. If initially there were **more equations than unknowns,** the final stage of the process will result in more than one equation in one unknown. Such a set of equations, as already shown in § 54, is **inconsistent.**

59. In the examples that follow, we restrict ourselves to the case in which there are as many equations as unknowns, so that the solution is unique.

Example 8. Solve
$$4x+3y-4z=1 \quad . \quad . \quad . \quad . \quad (1)$$
$$3x-5y-5z=4 \quad . \quad . \quad . \quad . \quad (2)$$
$$2x+7y+3z=5 \quad . \quad . \quad . \quad . \quad (3)$$

We shall first eliminate x between equations (1) and (2) by the method of equalizing the coefficients.

(1)×3 $\qquad 12x+9y-12z=3,$

(2)×4 $\qquad 12x-20y-20z=16.$

Subtracting, $\qquad 29y+8z=-13 \quad . \quad . \quad . \quad . \quad (4)$

We next eliminate x between equations (1) and (3).

(1) $\qquad 4x+3y-4z=1,$

(3)×2 $\qquad 4x+14y+6z=10.$

Subtracting, $\qquad -11y-10z=-9 \quad . \quad . \quad . \quad . \quad (5)$

We now eliminate one unknown between equations (4) and (5). As the L.C.M. of the coefficients of z is smaller than the L.C.M. of the coefficients of y, we shall eliminate z.

$(4)\times 5$	$145y+40z=-65,$
$(5)\times 4$	$-44y-40z=-36.$
Adding,	$101y=-101,$
	$\therefore\quad y=-1.$

Substituting this value of y in one of the equations containing y and the letter last eliminated, namely z, that is in equation (4) or (5), we obtain $z=2$.

Substituting the values, $y=-1$, and $z=2$, in any of the original equations, we obtain $x=3$.

Thus the solution is $x=3,\quad y=-1,\quad z=2.$

The pupil should verify that these values satisfy **each** of the three original equations.

Note.—It is usual to give the values of all the unknowns together, in alphabetical order, at the end of the working.

Example 9. Solve

$$3x-y+2z=4 \quad \ldots \quad (1)$$
$$7x-5y+3z=8 \quad \ldots \quad (2)$$
$$5x+7y+11z=2 \quad \ldots \quad (3)$$

It will be noticed that as the coefficient of y is unity in one of the equations, the multiplications necessary to eliminate y twice are easier than those involved in the elimination of either of the other two unknowns; hence we shall eliminate y.

$(1)\times 5$	$15x-5y+10z=20,$	
(2)	$7x-5y+3z=8.$	
Subtracting,	$8x+7z=12$	(4)
$(1)\times 7$	$21x-7y+14z=28,$	
(3)	$5x+7y+11z=2.$	
Adding,	$26x+25z=30$	(5)

We next eliminate z between equations (4) and (5).

$(4)\times 25$	$200x+175z=300,$
$(5)\times 7$	$182x+175z=210.$
Subtracting,	$18x=90,\quad \therefore\quad x=5.$

Substituting this value of x in equation (4), we obtain

$$z=-4 \quad . \quad . \quad . \quad \text{(A)}$$

Substituting the values $x=5$ and $z=-4$ in equation (1) . (B)
we obtain $y=3$.

Thus the solution is $x=5$, $y=3$, $z=-4$.

Note.—In line (A) the value of z could have been found by substituting in either of the equations (4) or (5), but substitution in (4) involved less arithmetic. Similarly in line (B), the substitution could have been made in equations (1), (2) or (3), but equation (1) was chosen as that involving less arithmetical work.

Example 10. Solve $3x-2y-4z+3=5x+3y-3z+18$
$=x-2y+2z+1=6x+y-3z+8$.

This is equivalent to three separate equations, since we may write the first expression equal to the second, third or fourth expressions, thus involving the equality of any two of the expressions. The necessary condition of § 58, case 1, three equations for the three unknowns, is therefore satisfied. Thus we may write:

$$3x-2y-4z+3=5x+3y-3z+18,$$

$$\therefore \quad 2x+5y+z=-15 \quad . \quad . \quad . \quad . \quad (1)$$

$$3x-2y-4z+3=x-2y+2z+1,$$

$$\therefore \quad 2x-6z=-2,$$

$$\therefore \quad x-3z=-1 \quad . \quad . \quad . \quad . \quad (2)$$

$$3x-2y-4z+3=6x+y-3z+8,$$

$$\therefore \quad 3x+3y+z=-5 \quad . \quad . \quad . \quad . \quad (3)$$

It will be observed that y does not occur in equation (2). Thus we need only eliminate y between equations (1) and (3) to obtain two equations in x and z.

(1)×3 $\qquad 6x+15y+3z=-45$,

(3)×5 $\qquad 15x+15y+5z=-25$.

Subtracting, $\qquad -9x-2z=-20$,

(2)×9 $\qquad 9x-27z=-9$.

Adding, $\qquad -29z=-29$, $\therefore z=1$.

Substituting in equation (2), $x=2$.

Substituting in equation (1), $y=-4$.

Thus the solution is $x=2$, $y=-4$, $z=1$.

The questions in Exercise 11c, page 134, should now be attempted.

Literal Simultaneous Equations

60. Example 11. Find the values of x, y, and z from the equations:

$$2x-2y+z=0 \quad . \quad . \quad . \quad . \quad (1)$$
$$2bx+y-az=a+b \quad . \quad . \quad . \quad (2)$$
$$3x-2by+bz=3a-2ab \quad . \quad . \quad . \quad (3)$$

These are literal equations (see § 45), and the values of x, y, and z can be found in terms of a and b.

The unknown easiest to eliminate is y.

(2)×2 $\quad 4bx+2y-2az=2a+2b.$

(1) $\quad 2x-2y+z=0.$

Adding, $\quad 4bx+2x-2az+z=2a+2b.$

This can be written $\quad (4b+2)x+(-2a+1)z=2a+2b \quad . \quad (4)$

(1)×b, $\quad 2bx-2by+bz=0,$

(3) $\quad 3x-2by+bz=3a-2ab.$

Subtracting, $\quad 2bx-3x=2ab-3a.$

This can be written $\quad (2b-3)x=a(2b-3),$

$\therefore \quad x=a.$

Substituting in equation (4),

$$(4b+2)a+(-2a+1)z=2a+2b,$$
i.e. $\quad 4ab+2a+(-2a+1)z=2a+2b,$

$\therefore \quad (-2a+1)z=-4ab+2b.$

This can be written $\quad (-2a+1)z=2b(-2a+1),$

$\therefore \quad z=2b.$

Substituting the values $\quad x=a$, $z=2b$ in equation (1),

$$2a-2y+2b=0,$$
$\therefore \quad 2y=2a+2b, \quad \therefore \quad y=a+b.$

Thus the solution is $\quad x=a, \quad y=a+b, \quad z=2b.$

The questions in Exercise 11d, page 136, should now be attempted.

Exercise 11a

Solve, using Method 1 :

1. $x+y=8$, $x-y=4$.
2. $x+y=17$, $x-y=7$.
3. $x-y=2$, $x+y=9$.
4. $y-x=5$, $x+y=10$.
5. $x+y=6$, $x-y=0$.
6. $x+2y=6$, $y=x$.

Solve, using Method 2 :

7. $y+2x=4$, $y+x=0$.
8. $2x+y=4$, $x+y=3$.
9. $2x-y=5$, $x+2y=5$.
10. $x-2y=1$, $2x+y=12$.
11. $x+2y=4$, $y-x=5$.
12. $x-2y=6$, $x-y=5$.

Solve, using Method 3 :

13. $x+2y=-1$, $2x+y=4$.
14. $x-2y=-1$, $2x+y=-7$.
15. $2x-3y=5$, $x-2y=1$.
16. $3x-8y=2$, $x-2y=2$.
17. $2x+3y=-5$, $3x+y=3$.
18. $x-3y=3$, $2x-5y=4$.

Solve :

19. $3x+y=6\frac{1}{2}$, $2x-3y=8$.
20. $3x-2y=6$, $2x-5y=-1\frac{1}{2}$.
21. $2\frac{1}{2}x-3y=4$, $2x-1\frac{1}{2}y=5$.
22. $1\frac{1}{2}x-3\frac{1}{2}y=-3$, $3x-5y=0$.
23. $\frac{2}{3}x-\frac{5}{6}y=7$, $x+\frac{2}{3}y=-1$.
24. $\frac{5}{6}x-\frac{3}{4}y=2$, $\frac{1}{3}x+y=6$.
25. $x+3y=1$, $5y+x+3=0$.
26. $2x+5y+5=0$, $3x+2y-9=0$.
27. $4x+3y+1=0$, $2y+3x+2=0$.
28. $\frac{3}{4}x-2y=9$, $2x+\frac{5}{8}y-3=0$.
29. $2x+1=3y$, $6y+3=3x$.
30. $2y-11=x$, $3x+5=2y$.
31. $3x+9=-4y$, $2x-16=y$.
32. $\frac{1}{4}x-3=4y$, $3y-1=\frac{1}{2}x$.
33. $11x+7y=19$, $5x-4y=23$.
34. $9x-13y=5$, $7x+5y=19$.
35. $17x+12y=9$, $13x+11y=16$.
36. $5x+19y=1$, $11x+23y=21$.
37. $15x+19y=8$, $13x+17y=0$.
38. $23x-12y=25$, $21x-13y+16=0$.
39. $17x+15y=15$, $25x+23y=7$.
40. $19x+27y+9=0$, $16x+25y=6$.

Solve, making use of the principle in the note to Example 4 § 56 :

41. $15x+7y=-6$, $20x+11y=-13$.

42. $55x-38y=5$, $33x-23y=1$.

43. $11x+28y=10$, $13x+35y=11$.

44. $32x+45y+8=0$, $47x+63y-13=0$.

45. $56x+31y=1$, $63x+34y=9$.

46. $52x-17y=1$, $39x-14y=-3$.

47. $72x-35y=1$, $5x+21y=23\frac{1}{2}$.

48. $13x+57y=3$, $21x+95y+1=0$.

Solve, correct to two decimal places :

49. $19x-13y=5$, $47x+21y=83$.

50. $17x+15y=123$, $26x-17y+21=0$.

Exercise 11b

Solve the equations :

1. $\frac{1}{x}+\frac{1}{y}=4$, $\frac{1}{x}-\frac{1}{y}=2$.

2. $\frac{1}{x}+\frac{1}{y}=\frac{2}{3}$, $\frac{1}{y}-\frac{1}{x}=\frac{4}{3}$.

3. $\frac{1}{x}+\frac{1}{y}=\frac{5}{12}$, $\frac{4}{x}-\frac{3}{y}=\frac{1}{2}$.

4. $\frac{2}{x}+\frac{5}{y}=1$, $\frac{14}{x}+\frac{15}{y}=5$.

5. $\frac{3}{x}-\frac{2}{y}=6$, $\frac{7}{x}+\frac{12}{y}=4$.

6. $\frac{8}{x}+\frac{3}{y}=8$, $\frac{10}{x}+\frac{9}{y}=3$.

7. $\frac{16}{x}+\frac{7}{y}=5$, $\frac{11}{x}+\frac{5}{y}=3\frac{1}{2}$.

8. $\frac{13}{x}+\frac{4}{y}=2\frac{1}{2}$, $\frac{12}{x}+\frac{7}{y}=-1$.

9. $2x+5=xy$, $3x+2xy=45$.

10. $\frac{2\cdot1}{2x}+\frac{1\cdot2}{5y}=0\cdot33$, $\frac{1\cdot5}{x}-\frac{1}{2y}=0\cdot05$.

11. $3x-\frac{8}{y}=11$, $2x-\frac{9}{y}=5\frac{1}{2}$.

12. $4x+\frac{3}{y}=1$, $5x+\frac{4}{y}=-1$.

13. $\frac{3}{x}+5y=8$, $\frac{5}{x}+9y=10$.

14. $\frac{10}{x}-3y+12\frac{1}{2}=0$, $2y-9=\frac{4}{x}$.

15. $5xy-13y=8$, $3xy-14y+20=0$.

16. $3(x+1)+2(y+1)=2,$
$2(x+3)-3(y+1)=2.$

17. $5(x+3)+3(y+2)=7,$
$4(x+2)+11(y-1)=3.$

18. $5x-6y+2=x+4y-1=2y-x+8.$

19. $7(x-\frac{1}{2})-8(y+\frac{1}{2})=2,$
$3(x+\frac{1}{2})-5(y+1)=-1.$

20. $\frac{1}{4}(x+3)+\frac{1}{3}(y+5)=\frac{1}{3}(x-2)-\frac{1}{2}(y-2)=3.$

21. $\frac{1}{5}(x+8)-\frac{1}{2}(y-1)=6,$
$\frac{1}{6}(x+5)+\frac{1}{3}(y+2)=1.$

22. $7x+11y-9=2(x-2y-7)=5x+13y+5.$

23. $3(3x+4y)=2(7x+10y)=6.$

24. $\frac{1}{4}(2x+1\frac{1}{2})+7(y-\frac{1}{2})=5,$
$\frac{1}{5}(2x-\frac{1}{2})+\frac{1}{4}(2y+1)+1=0.$

25. $\frac{1}{3}(x+\frac{2}{3})-3(y+3)=4,$
$\frac{1}{5}(3x+3)+\frac{1}{7}(y-3)=1.$

26. $17x+5y-2=23x+6y-5=2x-3y-11.$

27. $23x-6y=11x-3y+3=x+3y-19.$

28. $3(3x-2y+5)+2(4x-3y+1)=1,$
$11(2x-y+1)-9(5x-3y+3)=4.$

29. $8(4x+3y+1)+7(x+2y+1\frac{1}{2})=2,$
$7(5x+4y+\frac{1}{2})-3(3x-\frac{1}{2}y-7)=1.$

30. $5(5x-6y+3)=4(7x-8y+1)+3,$
$7(11x-12y-3)=5(9x-10y+1)-4.$

Exercise 11c

Solve the equations :

1. $x+y+z=6,$
$x+y-z=2,$
$2x+3y-5z=1.$

2. $x-y+z=0,$
$x+2y-z=7,$
$2x+y+4z=-1.$

3. $3x+2y=7,$
$4x+3y-z=4,$
$3x+2y+z=13.$

4. $5x+2z=17,$
$8x-y+6z=26,$
$8x+3y-12z=24.$

5. $2x-3y+2z=2,$
$x+2y+z=1,$
$3x-5y+2z=6.$

6. $14x-2y+3z=12,$
$10x-3y+2z=2,$
$14x+y+3z=24.$

7. $4x+3y-4z=1,$
$5x+7y+z=10,$
$3x-2y-5z=1.$

8. $8x+y+2z=35,$
$2x-y+6z=21,$
$7x+y+2z=31.$

9. $8x-6y-z=41,$
$3x+2y=2(z+16),$
$x+z=17-8y.$

10. $3x-4y-2z=11,$
$4x+y+5z=3,$
$6x-9y-2z=11.$

11. $8x-4y+z=-2,$
$16x-7y-3z=-11,$
$14x+9z=8(y+1).$

12. $3x+2y+3z=17,$
$4x+5y+2z=5,$
$5x+2y-4z=8.$

13. $9z-x=14(y+2),$
$x+2z=10+3y,$
$x+3z=2(11-2y).$

14. $3x+y+12z=33,$
$3(3x+2z)=8y-9,$
$2x+3y+10z-34=0.$

15. $\frac{x}{4}-\frac{2y}{5}+z=15,$

$2y-8=\frac{z}{7},$

$z-\frac{x}{4}=11.$

16. $\frac{1}{4}x+\frac{1}{36}y+z=16,$
$\frac{1}{6}(x-\frac{1}{3}y)=6-\frac{1}{3}z,$
$\frac{1}{9}x-1=2(z-\frac{1}{8}y).$

17. $3x+y+z=20,$

$\frac{1}{2}(x-3y)=\frac{2z}{3}-5,$

$\frac{3x}{2}+\frac{2z}{3}=10.$

18. $\frac{y}{3}+z-\frac{x}{2}=\frac{x}{2}-y+z=\frac{1}{2}\left(\frac{x}{2}+\frac{y}{3}-2z+8\right)=3.$

19. $2(\frac{1}{3}x+\frac{1}{4}z-2)=\frac{1}{3}x+z-12=\frac{2}{3}(\frac{1}{3}x+\frac{9}{4}z-36)=y.$

20. $\frac{1}{2}(x+y)+2z=48,$
$y+\frac{2}{3}z=\frac{1}{4}(57-x),$
$\frac{1}{2}x+\frac{4}{3}z=38\frac{1}{2}.$

21. $2x+9y-4z=13{\cdot}7,$
$2(2z-x)=4{\cdot}3-3y,$
$4z+2x-3y=10{\cdot}9.$

22. $6a+2b-d=11,$
$4a+b-2c-2d=5,$
$2a-b+4c+d=4,$
$2a-3b+2c+d=10.$

23. Show that the equations

$$5x-2y=3, \qquad 4y-5=10x$$

are inconsistent.

24. Show that the equations

$$4x+7y-5z=8,$$
$$3x-2y+4z=5,$$
$$x-20y+22z+1=0,$$

are not independent.

25. What value must a have in order that the equations

$$3x+5y=4,$$
$$2x-3y=9,$$
$$ax+7y+1=0,$$

may be consistent?

Exercise 11d

Solve the equations :

1. $3x+4y=a,$
$2x-3y=b.$

2. $cx+3y=2d,$
$3cx-2y=d.$

3. $5x-4my=7n,$
$2x+3my=5n.$

4. $ax+by=a,$
$ax-by=b.$

5. $3x=2y,$
$ax+by=c.$

6. $4ax=3by,$
$5ax-2by=7ab.$

7. $3nx+2my+mn=0,$
$4nx+5my=8mn.$

8. $5x=2qy,$
$px+qy=r.$

9. $4gx+fy=3fg,$
$7gx+3fy+fg=0.$

10. $x+y=a+b,$
$ax+by=a^2+b^2.$

11. $\dfrac{9a}{x}-\dfrac{4a}{y}=1,$
$\dfrac{6a}{x}+\dfrac{2a}{y}=3.$

12. $mx=3(m^2-y),$
$nx-5y=3mn.$

13. $\dfrac{8a}{x}+\dfrac{2b}{y}=1,$
$\dfrac{14a}{x}-\dfrac{5b}{y}=6.$

14. $cx=2cd-dy,$
$dx+c^2=cy+d^2.$

15. $4x+p(y+q)=2p^2,$
$7x-q(y-2p)=q^2.$

16. $bx+2y=b^2,$
$ax-y+2a^2=0.$

17. $lx-my=2(l^2-m^2),$
$x-2y+3m=0.$

18. $\dfrac{x}{a-b}+\dfrac{y}{a+b}=0,$
$ax-by=a^2+b^2.$

19. $x+y+z=a-1,$
$ax=2z,$
$2ax+ay-z=0.$

20. $x+y+z=0,$
$ax+2y=a^2,$
$3y-az=a^2.$

CHAPTER XII

Problems Leading to Simultaneous Equations

61. In § 58, 1, we showed that to obtain a set of values for the unknowns used in simultaneous equations, we require as many equations as there are unknowns. Hence in solving problems there must be as many conditions, each giving rise to an **independent** equation, as there are unknowns. In forming the equations, it will of course be necessary, as pointed out in § 49, to specify clearly what each of the unknowns represents, and also the units in terms of which they are expressed.

Example 1. If 3 cigars and 10 cigarettes cost 2s. 2d., while 5 cigars and 50 cigarettes cost 7s. 1d., find the price of a cigar and a cigarette.

Working in pence:

Denote the price of 1 cigar by x pence, and of 1 cigarette by y pence.

From the first condition,

$$3x+10y=26 \quad . \quad . \quad . \quad . \quad (1)$$

From the second condition,

$$5x+50y=85 \quad . \quad . \quad . \quad . \quad (2)$$

$$(1)\times 5 \qquad 15x+50y=130 \quad . \quad . \quad . \quad . \quad (3)$$

$$(3)-(2) \qquad 10x=45,$$

$$\therefore \quad x=4\tfrac{1}{2}.$$

Substituting this value of x in (1), we obtain

$$y=1\tfrac{1}{4}.$$

Thus the price of a cigar is $4\frac{1}{2}$d. and of a cigarette $1\frac{1}{4}$d.

Verification. 3 cigars and 10 cigarettes cost

$3\times 4\frac{1}{2}$d.$+10\times 1\frac{1}{4}$d.$=$1s. $1\frac{1}{2}$d.$+$1s. $0\frac{1}{2}$d.$=$2s. 2d.

5 cigars and 50 cigarettes cost

$5\times 4\frac{1}{2}$d.$+50\times 1\frac{1}{4}$d.$=$1s. $10\frac{1}{2}$d.$+$5s. $2\frac{1}{2}$d.$=$7s. 1d.

Example 2. Find the numerator and denominator of a fraction which becomes equal to $\frac{1}{2}$ if the numerator is doubled and the denominator is increased by 9, and which becomes equal to $\frac{1}{4}$ if the numerator is halved and the denominator is decreased by 1.

Denote the numerator of the fraction by x, and the denominator by y.

From the first condition,

$$\frac{2x}{y+9}=\frac{1}{2}.$$

It is generally convenient to write any equation formed from a condition of a problem in its simplest form before numbering it and proceeding to the next condition.

Thus, multiplying both sides by $2(y+9)$,

$$4x=y+9,$$

$$\therefore\quad 4x-y=9 \quad . \quad . \quad . \quad . \quad (1)$$

From the second condition,

$$\frac{\frac{1}{2}x}{y-1}=\frac{1}{4}.$$

Multiplying both sides by $4(y-1)$,

$$2x=y-1,$$

$$\therefore\quad 2x-y=-1 \quad . \quad . \quad . \quad . \quad (2)$$

$(1)-(2)$ $$2x=10,$$

$$\therefore\quad x=5.$$

Substituting this value of x in (2), $y=11$.

Thus the numerator and the denominator of the fraction are 5 and 11 respectively.

Verification.

$$\frac{2\times 5}{11+9}=\frac{10}{20}=\frac{1}{2}$$

$$\frac{\frac{1}{2}\times 5}{11-1}=\frac{2\frac{1}{2}}{10}=\frac{1}{4}.$$

Example 3. A butcher buys 10 pigs and 24 sheep for £143. Had he bought 5 more pigs, but paid 10s. a head less, and 4 fewer sheep at 3s. a head less, he would have spent £9 more in all.

What was the price of each pig and sheep ?

Working in £'s:

Denote the original price of a pig by £x, and the price of a sheep by £y.

$$\therefore\quad 10x+24y=143 \quad . \quad . \quad . \quad (1)$$

In the second case each pig costs £$(x-\frac{1}{2})$, and each sheep costs £$(y-\frac{3}{20})$, and the numbers purchased are 15 and 20 respectively. The total cost is £152.

$$\therefore \quad 15(x-\tfrac{1}{2})+20(y-\tfrac{3}{20})=152,$$
$$\therefore \quad 15x-7\tfrac{1}{2}+20y-3=152.$$

Multiplying throughout by 2,

$$30x-15+40y-6=304,$$
$$\therefore \quad 30x+40y=325.$$

Dividing throughout by 5,

$$6x+8y=65 \quad . \quad . \quad . \quad . \quad (2)$$

$$(2)\times 3 \qquad 18x+24y=195,$$
$$(1) \qquad 10x+24y=143.$$

Subtracting, $\quad 8x=52,$

$$\therefore \quad x=6\tfrac{1}{2}.$$

Substituting in equation (2), $\quad y=3\frac{1}{4}$.

Thus each pig costs £6, 10s., and each sheep costs £3, 5s. The student should verify that these values satisfy *each* condition of the question.

Example 4. A person invests a sum of money partly at 4 per cent. and partly at $2\frac{1}{2}$ per cent., and obtains a total return of £8, 15s. Had he invested at $2\frac{1}{2}$ per cent. the portion which he invested at 4 per cent., and at 4 per cent. the portion which he invested at $2\frac{1}{2}$ per cent., his return would have been 7s. 6d. more. How much did he invest altogether ?

Working in £'s:

Denote the amount invested at 4 per cent. by £x, and the amount invested at $2\frac{1}{2}$ per cent. by £y.

The return on £x at 4 per cent. is £$\frac{4}{100}x$. The return on £y at $2\frac{1}{2}$ per cent. is £$\frac{2\frac{1}{2}}{100}y$. Now the total return is £8, 15s., that is £$8\frac{3}{4}$.

$$\therefore \quad \frac{4}{100}x+\frac{2\frac{1}{2}}{100}y=8\tfrac{3}{4},$$

$$\therefore \quad \frac{x}{25}+\frac{y}{40}=8\tfrac{3}{4}.$$

Multiply throughout by 200,

$$8x+5y=1750 \quad . \quad . \quad . \quad (1)$$

In the second case he invests £x at $2\frac{1}{2}$ per cent. and £y at 4 per cent., and his total return is £9, 2s. 6d., that is £$9\frac{1}{8}$.

$$\therefore \quad \frac{2\frac{1}{2}}{100}x+\frac{4}{100}y=9\frac{1}{8},$$

$$\therefore \quad \frac{x}{40}+\frac{y}{25}=9\frac{1}{8}.$$

Multiplying throughout by 200,

$$5x+8y=1825 \qquad . \quad . \quad . \quad (2)$$

(1)×8 $\qquad 64x+40y=14000,$

(2)×5 $\qquad 25x+40y=9125.$

Subtracting, $\qquad 39x=4875,$

$$\therefore \quad x=125.$$

Substituting in equation (1), $y=150$.

Thus he invested altogether £125+£150, that is £275.

Note.—The answer to a problem must be given in the form asked for. It is not sufficient merely to record the values of the unknowns used in forming the equations.

The questions in Exercise 12a, page 145, should now be attempted.

Harder Problems

62. To facilitate the solution of a problem, it is advisable to use as few unknowns in the working as possible. The following examples, which are of a somewhat more difficult nature, will illustrate how this may be achieved.

Example 5. A lady buys bananas at 4 for 3d. and pears at $2\frac{1}{2}$d. each, and obtains a certain sum as change out of the 5s. which she tenders. Had she bought half as many bananas again, her change would have been 3d. If, however, she had bought two more bananas and one more pear than originally, her change would have been two-thirds of its first value. (*a*) Find how many bananas and pears were bought. (*b*) How many of each should she buy in order to obtain a total of 66 pieces of fruit in spending the whole 5s.?

(*a*) Working in pence:

Denote the number of bananas purchased by x, and the number of pears by y.

In the second case, the numbers purchased are $1\frac{1}{2}x$ and y respectively.

Hence the total outlay is

$$[\tfrac{3}{4}(1\tfrac{1}{2}x)+2\tfrac{1}{2}y] \text{ pence.} \quad (\because \text{ one banana costs } \tfrac{3}{4}\text{d.})$$

Now the change out of 5s., that is 60 pence, is 3d.

$$\therefore \quad 60-\tfrac{3}{4}(1\tfrac{1}{2}x)-2\tfrac{1}{2}y=3.$$

Multiplying throughout by 8,

$$480-9x-20y=24,$$

$$\therefore \quad 9x+20y=456 \quad . \quad . \quad . \quad . \quad (1)$$

The original outlay was $[\frac{3}{4}x+2\frac{1}{2}y]$ pence.

Hence the original change out of 5s. was

$$[60-\tfrac{3}{4}x-2\tfrac{1}{2}y] \text{ pence.}$$

Had she bought two more bananas and one more pear, her outlay would have been $[\frac{3}{4}(x+2)+2\frac{1}{2}(y+1)]$ pence.

Hence her change would have been

$$[60-\tfrac{3}{4}(x+2)-2\tfrac{1}{2}(y+1)] \text{ pence.}$$

Now the question states that this is two-thirds of its original value, $\therefore \quad 60-\frac{3}{4}(x+2)-2\frac{1}{2}(y+1)=\frac{2}{3}(60-\frac{3}{4}x-2\frac{1}{2}y)$.

Multiplying throughout by 12,

$$720-9(x+2)-30(y+1)=8(60-\tfrac{3}{4}x-2\tfrac{1}{2}y),$$

i.e. $\quad 720-9x-18-30y-30=480-6x-20y,$

$$\therefore \quad 3x+10y=192 \quad . \quad . \quad . \quad . \quad (2)$$

(2)×3 $\quad 9x+30y=576,$

(1) $\quad 9x+20y=456.$

Subtracting, $\quad 10y=120, \quad \therefore \quad y=12.$

Substituting in (2), $\quad x=24.$

Thus she bought 24 bananas and 12 pears.

(*b*) Suppose she buys n bananas.

Then she must purchase $(66-n)$ pears.

Now these together cost exactly 5s., that is 60 pence,

$$\therefore \quad \tfrac{3}{4}n+2\tfrac{1}{2}(66-n)=60.$$

Multiplying throughout by 4,

$$3n+660-10n=240,$$

$$\therefore \quad 7n=420, \quad \therefore \quad n=60.$$

Thus she must purchase 60 bananas and 6 pears.

Note.—In a problem of this type involving two separate questions, it is frequently simpler to solve one part independently, and then to use the results obtained to make the solution of the other part easier. This reduces the number of unknowns.

It will also be noted that in the first part of the question the use of a letter for the original change has been avoided.

Example 6. X can run at the rate of 9 yds. per second. In a race between X and Y, X wins by 66 ft. 8 ins. Had Y received 6 seconds start, Y would have won by 30 yds. In the latter case, if they continue running at the same rates as before, find how long it will take for X to overtake Y.

Working in feet, and feet per second. (See Note (1).)

Denote Y's speed by p ft. per sec., and the length of the race by q ft. X runs at the rate of 9 yds. per sec., that is 27 ft. in a second.

Hence the time taken for X to run q ft. is $\frac{q}{27}$ secs.

During this time Y runs a distance of 66 ft. 8 ins. less than q ft., that is $(q-66\frac{2}{3})$ ft.

Now Y's speed is p ft. per sec.

Thus in $\frac{q}{27}$ seconds Y runs $p\times\frac{q}{27}$ ft.

$$\therefore \quad p\times\frac{q}{27}=q-66\tfrac{2}{3},$$

$$\therefore \quad pq=27(q-66\tfrac{2}{3}),$$

$$\text{i.e.} \quad pq=27q-1800 \quad . \quad . \quad (1)$$

The time taken for Y to complete the distance of q ft. is $\frac{q}{p}$ secs.

Hence, in the second case, X runs for $\left(\frac{q}{p}-6\right)$ secs.

Now in this time he runs 30 yds. less than q ft., that is $(q-90)$ ft.

$$\therefore \quad 27\left(\frac{q}{p}-6\right)=q-90,$$

$$\text{i.e.} \quad 27\frac{q}{p}-162=q-90.$$

Multiplying throughout by p,

$$27q-162p=pq-90p,$$

$$\therefore \quad pq=27q-72p \quad . \quad . \quad . \quad (2)$$

Equating the values of pq in equations (1) and (2) . . (A)

$$27q-1800=27q-72p,$$

$$\therefore \quad 72p=1800, \qquad \therefore \quad p=25.$$

Thus Y runs at the rate of 25 feet per second.

Suppose X overtakes Y in n secs.
In n secs. X runs $27n$ ft.
In n secs. Y runs $25n$ ft.
Now Y is at present 90 ft. ahead of X,

$$\therefore \quad 27n=25n+90,$$

whence $$n=45.$$

Thus X overtakes Y in a further 45 seconds.

Notes.—(1) **The units must be compatible**; thus as the distances in the question were worked in terms of feet, the speeds must be in terms of feet per second and not yards per second.

(2) Since the term pq occurs in equations (1) and (2), these are not of the first degree. The phrase 'equating the values of pq' used in line (A) indicates the elimination by equating equivalent values of the term pq, as in Example 1, § 53. The resulting equation is linear.

(3) It is not always necessary to find the numerical values of all the unknowns used in the solution of a problem. Thus in this problem it has not been necessary to find the value of q.

Example 7. If a certain number be added to both the numerator and denominator of a given fraction its value becomes $\frac{1}{2}$, but if the same number be subtracted from both numerator and denominator its value becomes $\frac{1}{7}$. If, however, unity be added to both numerator and denominator the fraction becomes $\frac{2}{5}$. Find the fraction.

Denote the numerator of the required fraction by x, and the denominator by y.

Let z be the number added in the first case.

Then $$\frac{x+z}{y+z}=\frac{1}{2}.$$

Clearing of fractions,

$$2(x+z)=y+z,$$
$$\text{i.e.}\quad 2x+2z=y+z,$$
$$\therefore\quad 2x-y+z=0 \quad . \quad . \quad . \quad . \quad (1)$$

If z be subtracted, the fraction becomes $\frac{1}{7}$,

$$\therefore\quad \frac{x-z}{y-z}=\frac{1}{7},$$
$$\therefore\quad 7(x-z)=y-z,$$
$$\text{i.e.}\quad 7x-7z=y-z,$$
$$\therefore\quad 7x-y-6z=0 \quad . \quad . \quad . \quad (2)$$

If 1 be added, the fraction becomes $\frac{2}{5}$,

$$\therefore\quad \frac{x+1}{y+1}=\frac{2}{5},$$
$$\therefore\quad 5(x+1)=2(y+1),$$
$$\text{i.e.}\quad 5x+5=2y+2,$$
$$\therefore\quad 5x-2y=-3 \quad . \quad . \quad . \quad . \quad (3)$$

We have now to solve the simultaneous equations (1), (2) and (3).

Eliminating z between equations (1) and (2),

(1)×6 $\qquad 12x-6y+6z=0,$

(2) $\qquad 7x-y-6z=0.$

Adding, $\qquad 19x-7y=0 \quad . \quad . \quad . \quad . \quad (4)$

Eliminating y between equations (3) and (4),

(3)×7 $\qquad 35x-14y=-21,$

(4)×2 $\qquad 38x-14y=0.$

Subtracting, $\qquad -3x=-21, \quad \therefore \quad x=7.$

Substituting in equation (4), $\quad y=19.$

Thus the fraction is $\frac{7}{19}$.

Note.—The pupil should observe that although the question asks for the value of the fraction, this cannot be denoted by a single unknown, as its constituent numerator and denominator are both required by the conditions given in the example.

Referring again to Note (3) of the previous example, it will be seen that it has not been necessary to determine the value of z.

The questions in Exercise 12b, page 149, should now be attempted.

Exercise 12a

1. Find two numbers whose sum is 46 and whose difference is 12.
2. If one-sixth of the sum of two numbers is 8, and half their difference is 13, what are the numbers ?
3. The sum of two numbers is 25 and their difference is 7. What are the numbers ?
4. Find two numbers such that five times the first number added to three times the second number is 49, and the difference between three times the first number and five times the second number is 9.
5. Find two numbers such that one-seventh of their sum is equal to two-fifths of their difference, and twice the smaller is three less than the larger.
6. A number is made up of two digits whose sum is 11, and such that twice the tens digit minus the units digit is 1. What is the number ?
7. Find two numbers such that one-third of the greater added to twice the less is 10, and one-fifth of their sum is 2.
8. The difference of two numbers is 2, and the greater increased by 5 is equal to twice the less. What are the numbers ?
9. Find two numbers such that one-quarter of the greater exceeds one-sixth of the smaller by two, and twice the smaller increased by the greater is equal to 56.
10. If three-fourths of the smaller of two numbers is 6 less than two-thirds of the greater, and half the smaller added to half the greater is 1 more than the smaller, what are the numbers ?
11. A boy's age added to his sister's age is 18 years, while the difference in their ages is 4 years. How old are they ?
12. Three times the length of a table is equal to four times its breadth, while twice its length is less than three times its breadth by 1 foot. What are the dimensions of the table ?
13. A man carries seven shillings, divided between two pockets. If he had three times as much in one pocket and twice as much in the other he would have sixteen shillings. How much had he in each pocket ?
14. If two geese and three turkeys together cost £3, 8s. 6d., and five geese and four turkeys together cost £6, 0s. 6d., what is the price of each ?

15. A boat can be rowed 4 miles downstream and then 3 miles upstream in $\frac{3}{4}$ hour. If it is rowed 12 miles downstream and then 5 miles upstream it takes 1 hour 35 minutes. Find the rate of the stream, and what the rate of the boat would be in still water. (Denote speed downstream by x m.p.h., and speed upstream by y m.p.h.)

16. If two pounds of biscuits and one pound of cheese cost 4s. 8d., and one pound of biscuits and two pounds of cheese cost 4s. 4d., what is the price per pound of each ?

17. By cycling a certain distance and then walking I cover a distance of 60 miles. If I had cycled half as far and walked twice the previous distance, I would have travelled 48 miles. How far did I cycle, and how far did I walk ?

18. Find the fraction which becomes unity by increasing the numerator by 1, and becomes 2 by subtracting 5 from the denominator.

19. A train running from A to B passes into a bank of fog 10 miles from A, which causes it to continue at a quarter of its original speed, and thus it arrives $1\frac{1}{2}$ hours late at B. Had it run into fog 10 miles further on, it would have been only $\frac{3}{4}$ hour late. Find the original speed of the train, and the distance from A to B.

20. Find a fraction which becomes equal to $\frac{1}{2}$ if 1 is added to both enumerator and denominator, and equals $\frac{1}{3}$ if 1 is subtracted from both numerator and denominator.

21. If the numerator of a certain fraction is halved and the denominator is increased by $\frac{1}{2}$, the value of the fraction becomes $\frac{1}{3}$. If the numerator of the fraction is decreased by 1 and the denominator is increased by 5, the fraction becomes equal to $\frac{4}{9}$. What is the fraction ?

22. With a total outlay of £2, 4s. 2d. I can purchase 7 lbs. of coffee and 15 lbs. of tea, or 20 lbs. of coffee and 5 lbs. of tea. Find the price of each per pound.

23. The weekly wages bill for 7 men and 3 boys is £18, but when 5 men and 4 boys only were employed it was £3, 15s. less. What is the wage of a man and a boy ?

24. A line 36 inches long is divided into two parts. If the ratio of their lengths is expressed as a fraction, then on adding 1 to the numerator and subtracting 1 from the denominator, this fraction becomes unity. On subtracting 2 from numerator and adding 2 to denominator, the fraction becomes $\frac{1}{3}$. What are the lengths of the two parts ?

25. A man has £2, 5s. made up of half-crowns and shillings. If the total number of coins is 27, how many has he of each coin ?

26. If 3 gross of $\frac{3}{4}$-inch screws and 5 gross of $1\frac{1}{4}$-inch screws together cost 8s. 8d., while 7 gross of $\frac{3}{4}$-inch screws and 11 gross of $1\frac{1}{4}$-inch screws together cost 19s. 5d., what is the price of each size of screw per gross ?

27. By selling two kinds of chocolates, one at 2s. 8d. per box, and the other at 3s. 2d. per box, I receive £2, 17s. 4d. Had I sold 4 more boxes of the cheaper kind and twice as many boxes of the dearer kind I would have received £1, 16s. more. How many boxes of each kind did I sell ?

28. I employ 12 boys and 26 men and pay a weekly wage bill of £106. If I employ 3 more boys and 4 more men my wages bill increases by £17, 15s. What are the wages of a boy and a man ?

29. My total outlay on two rolls of cloth is £39. One of the rolls cost 13s. 6d. per yd., while the other cost 15s. per yd. I make a profit of 2s. per yd. on the first roll and 3s. 6d. per yd. on the second roll, my total profit being £7, 7s. 6d. Find the number of yards in each roll.

30. A cigarette-machine holds 52 packets of cigarettes, some in packets of 10 and some in packets of 20. If the total number of cigarettes is 760, how many packets of 10 and how many of 20 are there in the machine ?

31. The Arsenal play Brentford at football and Arsenal's score plus twice Brentford's is 13 goals. If Arsenal's score were doubled and Brentford's were two goals less, only 9 goals would have been scored. What was the result ?

32. A man can travel along a road for 15 miles by bus or trolley-bus. If he travels 6 miles by bus and the rest by trolley-bus it takes him $\frac{3}{4}$ of an hour. If he travels 12 miles by bus and the rest by trolley-bus he does the journey in 5 minutes less. What were the speeds of the bus and the trolley-bus ?

33. A board 10 feet 2 inches long has to be cut into 20 pieces, some of which are 9 inches long and others 4 inches long. If two inches is allowed for sawing waste and nothing is left over, how many of each length were cut ?

34. A party numbering 48 is made up of children and adults. The total cost of their railway tickets is £2, 12s. If the adult fare is 1s. 4d. and a child travels at half rate, how many adults and how many children were in the party ?

35. In a cricket match the first innings of both Smith's and Brown's elevens were declared, but the second innings of both sides were completed. The total runs scored in the match were 1038, and 31 wickets fell. Smith's side had an average of 33 runs per wicket, while Brown's side had an average of 34 per wicket. What was the result ?

36. Two customers buy the same quality of butter and bacon, and each tenders a £1 note. The first buys 8 pounds of butter and 5 pounds of bacon, and receives 1s. change, while the second buys 6 pounds of butter and 7 pounds of bacon, and receives 2s. 6d. change. What was the price per pound of the butter and bacon ?

37. I buy a number of history books at 2s. 6d. each, and a number of geography books at 3s. each, my total outlay being £7, 15s. If the history books were 3d. each dearer and the geography books 4d. each cheaper, my total outlay would be 4d. less. How many of each kind did I buy ?

38. A sum of money is invested partly at 5 per cent. and partly at 4 per cent., and yields an income of £22, 7s. If the amount invested at 5 per cent. had been invested at 4 per cent., and the amount at 4 per cent. invested at 5 per cent., the income would have been 15s. more. How much was invested altogether ?

39. A farmer sells potatoes at 4s. per cwt. and turnips at 9s. per cwt., and receives a total payment of £17, 3s. If he had sold his potatoes at 6d. per cwt. more and his turnips at 1s. per cwt. less, he would have received 11s. more. What weight of each did he sell ?

40. Two trains, A and B, approach each other from stations 288 miles apart. When A leaves 1 hour later than B they pass after B has travelled for 4 hours, but if A leaves $\frac{3}{4}$ of an hour before B they pass when B has travelled for 3 hours. What is the speed of the trains ?

41. By investing partly in $2\frac{1}{2}$ per cent. stock and partly in 3 per cent. stock, an income of £7, 13s. is obtained. If the sum invested in the 3 per cent. stock had been invested at $2\frac{1}{2}$ per cent., and the sum in the $2\frac{1}{2}$ per cent. stock invested in a $3\frac{1}{4}$ per cent. stock, the income would have been 2s. 6d. more. How much was originally invested at $2\frac{1}{2}$ per cent., and how much at 3 per cent. ?

42. If the number of births in a town falls by 5 per cent. and the number of deaths increases by 2 per cent., the net loss in population is 2. If instead the births increase by 10 and the deaths fall by 4 per cent., the net gain in population is 42. What are the number of births and deaths ?

Exercise 12b

1. I spend 5s. 8d. in buying oranges at $1\frac{1}{2}$d. each and pears at $2\frac{1}{2}$d. each. If I were to sell them all at 2d. each I should lose 4d. How many of each did I buy ?
2. I buy cigarettes at $\frac{1}{4}$d. each and cigars at 2d. each, my outlay being 18s. 9d. By selling the cigarettes at 10 for 3d. and the cigars at $2\frac{1}{2}$d. each, I make a profit of 4s. 2d. How many of each do I buy ?
3. A boat travels 24 miles upstream and then 36 miles downstream in 5 hours, but can go 36 miles upstream and 24 miles downstream in 5 hours 50 minutes. What was the rate of the boat in still water, and the rate at which the stream was flowing ?
4. A number consists of two digits, and is such that when increased by 18 the digits are reversed. What is the number if the sum of the digits is 8 ?
5. A number made up of two digits is equal to seven times the sum of its digits, and is such that when 18 is subtracted from it the digits are reversed. What is the number ?
6. A certain number consists of two digits, and is such that it is equal to four times the sum of its digits. If the number is increased by three times the tens digit the result is 60. What is the number ?
7. A boy who had been given 3s. 6d. with which to buy 10 blue and 16 red pencils should have received $\frac{1}{2}$d. change. By mistake the boy asked for 16 blue and 10 red pencils and found that his money was too little by one penny. Find the price of each pencil.
8. A shop sells its entire stock of oranges at 3 for 2d. and three-quarters of its stock of pears at $1\frac{1}{2}$d. each, the proceeds being £3, 4s. $9\frac{1}{2}$d. The following day the rest of the pears are sold at $1\frac{1}{4}$d. each, the proceeds being 7s. $3\frac{1}{2}$d. less than that obtained for the oranges alone on the previous day. How many oranges and pears were there at first ?

9. I buy algebra books at 3s. 6d. each and geometry books at 3s. each, and receive change for a £5 note. I then repeat the purchase, but increase the number of geometry books by two, and my change for a £5 note is 40 per cent. less than previously. Later I buy 25 per cent. more geometry books and two more algebra books than at first, and pay £5, 1s. How many algebra and geometry books do I buy altogether ?

10. A grocer has a 1-cwt. keg of salt, and from it fills all the 4-oz. and 6-oz. packets he possesses. Next day he obtains three-twentieths as many 4-oz. packets and one-ninth as many 6-oz. packets, and by filling these he uses up two-thirds of the salt left over in the keg. Finally, he again obtains one-ninth as many 6-oz. packets as at first, and by filling these he empties the keg. What weight of salt does he pack on the first day ?

11. A number consists of two digits and exceeds three times the sum of its digits by 7. If the number is increased by 17, the new number has its units digit one more than the old tens digit, and the tens digit two less than the old units digit. What is the number ?

12. The product of two numbers is six less than nine times the smaller number, but eight times the reciprocal of the smaller is one-ninth the sum of the larger number and four. What are the numbers ? (The reciprocal of x is $\frac{1}{x}$. See § 36.)

13. A number is made up of two digits such that four times the reciprocal of the tens digit plus three times the reciprocal of the units digit is 2. If five times the reciprocal of the tens digit minus twice the reciprocal of the units digit is $\frac{7}{12}$, what is the number ?

14. I have a box of lemons which I bought at $\frac{1}{2}$d. each, and a box of oranges which I bought at $\frac{3}{4}$d. each. If I sell the lemons at 5 for 2d. and the oranges at 7 for 6d. I gain 6d., but if I sell the lemons at 9 for 5d. and the oranges at 10 for 7d. I lose 2d. How many lemons and how many oranges did I buy ?

15. It takes 24 seconds for a boy, running at a constant rate, to run up and down a moving staircase 108 ft. long. If it takes him 12 seconds to reach the top and come down 36 ft., find
(*a*) the rate of the boy in ft. per second ;
(*b*) the time taken for a person to reach the top by walking up the staircase at half the rate of the staircase ?

16. A number consists of two digits. If 3 be added to the number, and the result divided by the sum of the digits, the quotient is 6. If 3 be subtracted from the number, and the result divided by the units digit, the quotient is 12. What is the number?

17. By selling 9 horses and buying 23 pigs a man increased the money he had by 20 guineas. If he had bought 5 horses and sold 7 pigs he would have decreased his money by £68. What is the price of a horse and the price of a pig?

18. A given number consists of two digits whose product is 24. If a certain number is added to the given number the sum is 44, but if this number is added to the sum of the digits of the given number the result is 17. What is the given number?

19. A bag contains £53, 10s. made up of 6d., 1s., and 2s. 6d. pieces. There are three times as many 6d. pieces as 2s. 6d. pieces. If half the number of 1s. pieces are taken out and replaced by as many pennies, the bag would be worth £41, 16s. 3d. How many 2s. 6d. pieces are there in the bag?

20. At a circus six times as many 6d. seats were sold as 2s. seats. The proceeds from the 1s. seats amounted to 5s. more than that from the 6d. seats. If the total takings came to £31, 17s., how many of each of the three seats were sold?

21. Five times the difference of two numbers equals half their product, while five times the sum of their reciprocals is $1\frac{1}{2}$. What are the two numbers?

22. The number of square feet in a rectangle is equal to five times the number of feet in the length, decreased by three times the number of feet in the breadth. If each side had been increased by 3 ft., the number of square feet in the area would have been seven times the number of feet in the original length, increased by 21. What are the dimensions of the rectangle?

23. Hughes and Evans are in a cycling race and Hughes wins by $193\frac{1}{3}$ feet, his rate being 22 ft. per sec. Had Hughes given Evans 5 secs. start, he would have beaten Evans by 86 ft. What was the length of the race, and what was Evans' rate?

24. A sum of money is divided among A, B and C. A's share is equal to half the sum of the shares of B and C, and B's share is 4s. 6d. less than half the sum of the shares of A and C. If A and B together receive £1, 7s., how much did each receive?

25. Two numbers are formed by the same two digits. If one be divided by the other the result is $\frac{2}{9}$, but if one be subtracted from the other the result is 63. Find the numbers.

26. A number consists of two digits. If the tens digit be added to the number and the result be divided by the sum of the digits, the quotient is 5. If 14 be subtracted from the number and the result doubled, the original number with its digits reversed is obtained. Find the number.

27. I pay £1, 9s. for 5 rabbits, 4 fowls and 1 goose, or £2, 3s. for 7 rabbits, 2 fowls and 3 geese. If, however, I buy 11 rabbits, 10 fowls and 8 geese, I pay £6. What is the price of a rabbit, a fowl, a goose ?

28. A three-digit number has the hundreds digit four times as much in excess of the tens as the units digit is in deficit of it. If the number is divided by the sum of its digits the result is 61, and if 495 be subtracted from the number, the result consists of the digits of the original number in reverse order. What is the number ?

29. At a garden party the entrance fee was 1s. for men, 6d. for women and 2d. for children In all, 1027 people paid for admission, and the amount from entrance fees was £21, 10s. If the men paid £1, 8s. more than the women, how many men, women and children paid ?

30. A man cycled at a certain uniform speed. Had he cycled 2 miles an hour slower the journey would have taken him $1\frac{1}{4}$ hours more, but if he had cycled 5 m.p.h. faster he would have saved 1 hour 40 minutes on the journey. How long was the journey, and what was his speed ?

31. A tank of water has a pipe always supplying it. The tank, if full at the beginning, is emptied by 40 people in 10 days, but takes 30 people 20 days to empty. By how many people would it be emptied in 40 days ? (Assume same quantity of water per person per day, and uniform inflow of water.)

32. A ship springs a leak, and a steady flow of water pours into a hold. Two pumps are set working, and at the end of 1 hour there is twice as much in the hold. Four more pumps are therefore put in action, and all the water is pumped out in another 2 hours. If the 6 pumps had been set going at first, how long would it have taken to remove the water ?

33. If a certain number is added to the numerator and denominator of a given fraction its value becomes $\frac{4}{7}$, but if

the same number be subtracted from both numerator and denominator its value becomes $\frac{1}{2}$. If, however, 7 is added to both numerator and denominator the fraction becomes $\frac{4}{5}$. What is the fraction ?

34. A trader buys a certain number of oranges at 6d. per score, and finds that after throwing away the bad ones and selling a number at 9 for 1s., he has taken 1s. less than his outlay. He now sells the remainder at $\frac{1}{2}$d. each, and thus recovers his outlay together with 3s. profit. If, however, after throwing out the bad ones, he had sold all his oranges at 3 for 2d., he would have made 2s. 10d. profit on his outlay. How many oranges does he buy, and how many are bad ?

35. Two towns, A and B, are 40 miles apart. A cyclist leaves A and a motorist leaves B, travelling towards each other, each leaving at 9 a.m. They pass each other at 10 a.m. The motorist stops at town A for 40 minutes, and then starts back at the same rate for town B. He overtakes the cyclist at noon, who is then as far from B as he was previously from A when first passed. What was the distance of cyclist from A when overtaken ?

36. A bill for £5, 14s. 6d. is paid in half-sovereigns, half-crowns, and shillings, 35 coins in all being given. The number of half-crowns added to the number of shillings is equal to four times the number of half-sovereigns. How many of each coin are there ?

37. Two trains travel on parallel tracks, the faster at 60 m.p.h. If they leave the station together the faster will pass a certain signal-box when the slower is 25 miles from it. If the faster train leaves $\frac{3}{4}$ hour after the slower, it will overtake it 75 miles beyond this signal-box. What is
(a) the speed of the slow train ?
(b) the distance of the signal-box from the station ?

38. A man buys 5 cups, 5 saucers and 7 plates, and pays 4s. 9$\frac{1}{2}$d. Had he bought 8 saucers and 8 plates it would have cost him 3$\frac{1}{2}$d. more than if he had bought 9 cups. If 8 cups and 3 saucers cost as much as 11 plates, find the cost of each article.

39. Find the values of a, b, c, so that the fraction $\dfrac{a+bx}{3+cx}$ shall have the values 2, 5, $\frac{1}{3}$ when $x=1$, $-\frac{1}{2}$, 3 respectively.

40. Of two fields, the first is 4 ft. longer and 1 yd. wider than the second, and the width of the second is $\frac{3}{4}$ the length of the first. If the area of the first is $50\frac{2}{3}$ sq. yds. more than the second, find the dimensions of each.

41. An aeroplane flies between two towns. If its speed is increased by 20 m.p.h., the time is reduced by 17 minutes; if its speed is reduced by 20 m.p.h., the time is increased by 19 minutes. Find the distance between the towns, and the normal speed.

42. A man's income is £34 less this year than last year, and was £80 less last year than the year before. The rate of income tax is 3d. more in the £ this year than last, and was 6d. more last year than the year before. If his income tax was the same for the three years, find this year's income and income tax.

43. A man uses 5 gallons of petrol and $\frac{1}{2}$ pint of oil on a journey, and the cost of these together with incidental expenses amounts to 12s. 5d. On a second journey he uses $3\frac{1}{2}$ gallons of petrol and $\frac{1}{4}$ pint of oil, and, the incidental expenses being 3 times as much as in the first case, the total cost is 17s. 5d. If 64 gallons of petrol and 58 pints of oil come to as much as 41 times his expenses on the first journey, find the latter, and also the cost of petrol and oil.

44. A man buys a case of tea, and sells at 2s. 8d. a lb. until he has recovered his outlay. He then sells the remainder at 2s. 6d. a lb., and finds that he has made $56\frac{1}{4}$ per cent. profit on his total outlay. How much per lb. did he pay for the tea, and what proportion of it was sold at the cheaper rate?

45. If 16 sheep were put to graze in a certain field, the grass would be consumed in 10 days; while if 17 sheep were put in, the grass would last 9 days. How many days would the grass last if 22 sheep were put in? (Assume that the grass grows at a uniform rate.)

46. The combined ages of a father and son are 147 years. The father is twice as old as the son was when the father was as old as the son is now. Find their present ages.

CHAPTER XIII

Graphs

63. A quantity whose value is not always the same is called a **variable**; for example, the daily number of visitors to the British Museum, the temperature, the height of a wave, the velocity of the wind are variables.

A quantity whose value does not change is called a **constant**; for example, the number of pence in a shilling, the ratio of the circumference of a circle to its diameter, the sum of the three angles of a triangle are constants.

Two variables may be connected in such a way that for a given value of one, there is some definite value of the other. Thus, referring to the variables enumerated above, if one variable be taken as the date, then for a given value of it the number of visitors to the British Museum has a definite value; or if one variable be the time, then for a given value of it the temperature at a given place has a definite value, and so on. These are examples of **connected variables.**

Straight Line Diagrams

64. It is convenient to tabulate sets of corresponding values of two connected variables. The following table gives the height of a boy on his successive birthdays:

Age in years . .	1	2	3	4	5	6
Height in inches .	28	33	37	40	43	45

It is easier to understand the significance of this data when it is set out in the form of a diagram.

The diagram is drawn in the following manner:

Two perpendicular lines are used, and are labelled to show what quantities are to be measured along them. In Fig. 10 these quantities are: Age in years, and Height in inches. Along the horizontal line, the years are marked at intervals of half an inch. Along the vertical line, a height of 2 inches is represented by one

small division. At each point on the horizontal line which represents one of the years, a vertical line is drawn whose length indicates the height of the boy at that age.

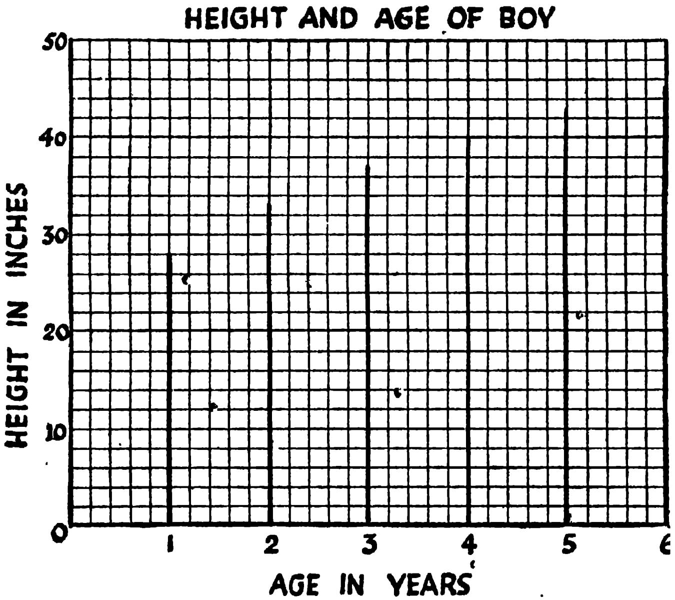

FIG. 10

Note.—It is not always necessary for the values shown on the diagram to start from zero. Since all the heights lie between 28 and 45 inches, it would have been sufficient if the vertical line had shown a **range** from 25 to 50 inches only. Had this been done, a height of 1 inch could have been represented by one small division, and the change in height would have been more easily seen.

65. In order that the tops of the vertical lines may be more readily seen, it is frequently useful to join consecutive tops by straight lines. If this is done and squared paper is used, the vertical lines themselves need not be emphasized.

When two sets of data are marked on the same diagram, it is essential that the connecting lines for each set of data be drawn.

Example 1. The mid-day temperatures at two towns, *A* and *B*, for the first half of May are recorded in degrees Fahrenheit in the following table:

Day .	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A .	76	75	74	71	73	73	74	77	72	76	78	79	77	77	78
B .	75	73	74	74	76	75	77	78	78	76	75	74	76	75	76

Represent both sets of data on a diagram, and use this to find:

(1) the dates in May when *A* was warmer than *B*;
(2) the dates in May when *A* was colder than *B*;
(3) the dates in May when *A* and *B* were at the same temperature;
(4) the date in May when *A* was the maximum amount warmer than *B*, and the number of degrees warmer;
(5) the date in May when *A* was the maximum amount colder than B, and the number of degrees colder.

The data is shown in Fig. 11, the full line representing town *A*, and the dotted line town *B*.

It is seen that all the temperatures are included in the range 70 to 80 degrees Fahrenheit.

On studying the diagram it can be seen that the answers to Questions 1 to 5 are as follows:

(1) 1st, 2nd, 11th, 12th, 13th, 14th, 15th.
(2) 4th, 5th, 6th, 7th, 8th, 9th.
(3) 3rd, 10th.
(4) 12th, 5 degrees Fahrenheit.
(5) 9th, 6 degrees Fahrenheit.

The questions in Exercise 13a, page 182, should now be attempted.

Smooth Curve Graphs

66. It is really not essential to draw the vertical lines at all. We can mark with dots the points where their tops would be.

A boy runs a mile race, and is timed over the furlong, quarter, half and full mile. The results are recorded in the following table:

Distance run in miles .	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1
Time in seconds . .	25	55	125	350

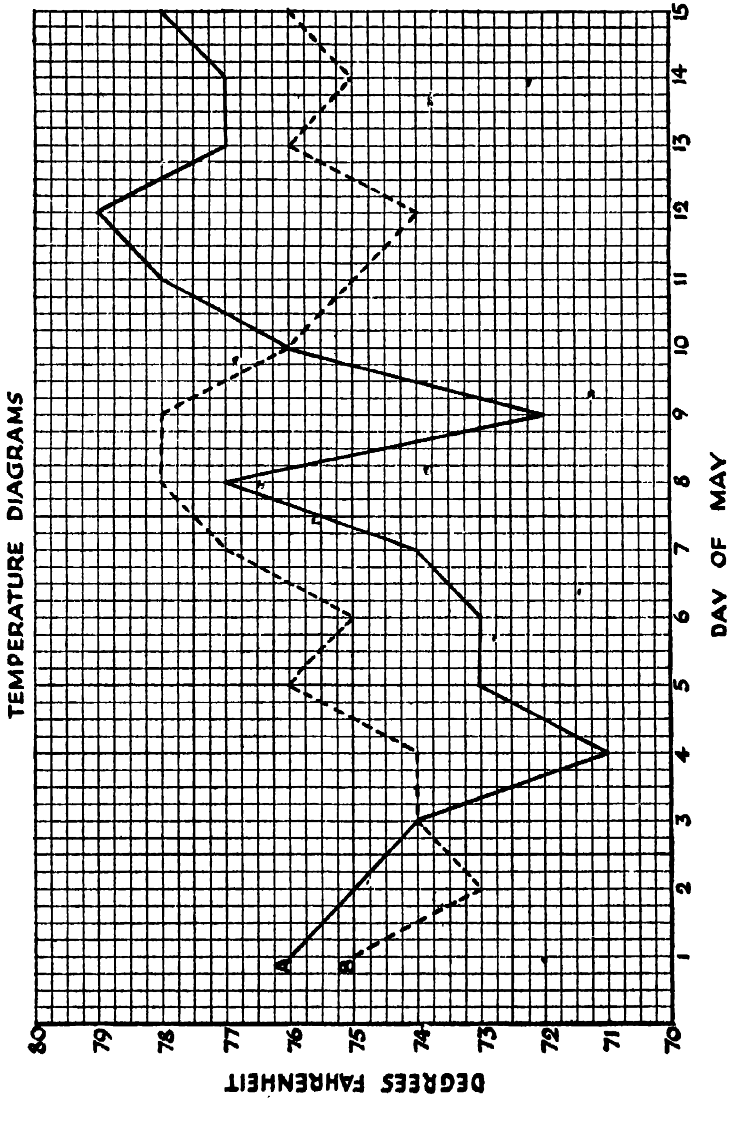
TEMPERATURE DIAGRAMS
DAY OF MAY
DEGREES FAHRENHEIT
A
B
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15
70 71 72 73 74 75 76 77 78 79 80

Fig. 11

The relation between the two connected variables, distance run and time, can be represented on squared paper shown as in Fig. 12. The distance run is recorded by length in a direction parallel to OX, the scale being that 1 inch represents $\frac{1}{4}$ mile.

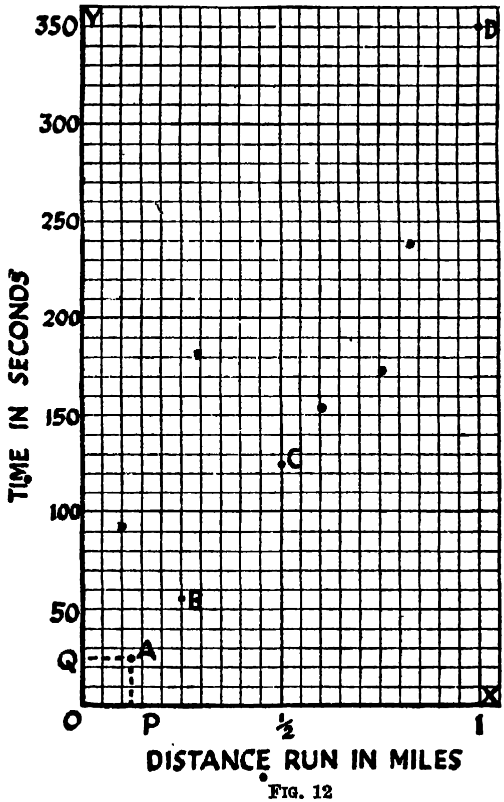

Fig. 12

Thus two and a half small divisions represent $\frac{1}{8}$ mile. The time is recorded by length in a direction parallel to OY, the scale being that 1 inch represents 100 seconds. Thus one small division represents 10 seconds.

Consider one of the marked points A, say, and through it draw lines AP, AQ parallel to OY and OX. It will be seen that OP

represents $\frac{1}{8}$ mile, OQ represents 25 seconds; so that the point marked A records the fact that the time for the $\frac{1}{8}$ mile is 25 seconds. It will thus be seen that given the values $\frac{1}{8}$ mile, 25 seconds, the points P and Q are known, and the position of the point A is determined. In the same way the points B, C, D record the other facts set out in the table.

Note.—In practice we omit the dotted lines AP, AQ, as these can be pictured mentally, even when they do not fall on the printed lines of the squared paper.

Interpolation

67. If the same boy is timed over a very large number of distances between $\frac{1}{8}$ mile and 1 mile in length, the times for two nearly equal distances would differ only slightly. When these are recorded on the diagram, the two points would lie very close together. If, therefore, we record on the diagram all the points for the very large number of distances and join them, we should obtain something approaching a smooth curve.

Now refer to Fig. 13. It will be seen that the points A, B, C, D are marked as in Fig. 12, and that a smooth curve has been drawn through them. This is the curve we should expect to obtain if the data for the very large number of distances mentioned above were available.

Consider the point K on this curve. It will be seen to correspond to a race of $\frac{3}{4}$ mile and a time of 225 seconds. It is reasonable to suppose that if the boy had been timed over a distance of $\frac{3}{4}$ mile, the time that would be recorded would be 225 seconds.

It will be seen that once we have drawn a smooth curve through the points obtained from the original data, we can find additional values by choosing other points on the curve, such as K. This process of finding values intermediate to those tabulated is called **interpolation.**

68. A curve drawn through the points obtained from the data connecting two quantities is called a **graph,** and the method of drawing it is called **plotting** the graph. Thus a graph is a curve which indicates the relation between two connected variables.

In a graph such as Fig. 13, ***OX* and *OY* are called the axes, and the point *O* is called the origin.** The position of any point on the graph is determined by the lengths of two lines, that is the lengths of the perpendiculars from the point to OY and OX.

These two lengths are called the **co-ordinates** of the point, sometimes distinguished by calling the former the **abscissa** and

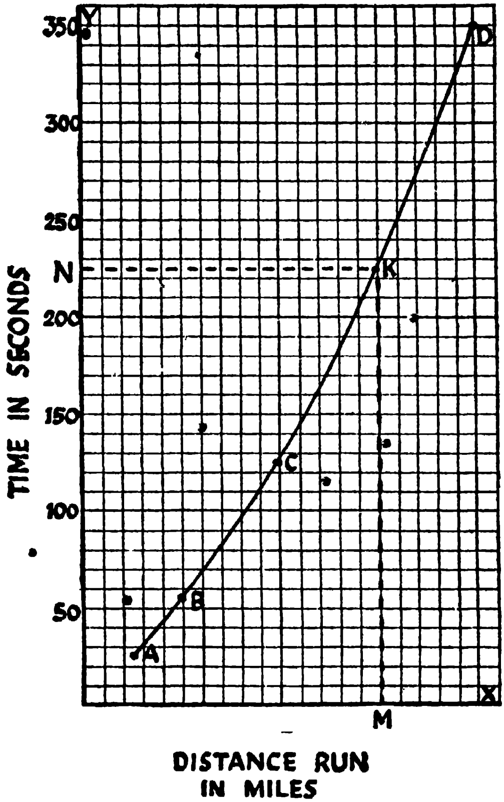

FIG. 13

the latter the **ordinate.** Thus in the case of the point K the abscissa is NK, and the ordinate is MK; these are equal to OM and ON respectively.

Axes and Scales

69. We see that a graph shows, in diagram, the change produced in one quantity by a change in another quantity with which it

is connected. These quantities we have already referred to as connected variables.

It is usually natural to think of one of these variables as that in which the changes are made; this we call the **independent variable.** The other, which has changes produced in it in consequence of its connection with the first, we call the **dependent variable.**

In drawing a graph, we represent the independent variable by distances measured from left to right (abscissa), and the dependent variable by distances measured vertically upwards (ordinate).

70. To find scales appropriate to the size of paper available, the following should be kept in mind. If either of the scales is too small, the graph cannot be drawn with accuracy, and thus reliable readings cannot be taken from it. If either of the scales is too large, it may not be possible to plot all the necessary values of the variables. To facilitate plotting the points and reading the values, scales in which 1 inch represents numbers such as 3, 7, 11 should be avoided. Instead, numbers such as 1, 10, 100, 0·1, 0·01, 2, 5, 0·2, 0·5, etc., should be chosen, the particular choice, of course, depending upon the range to be covered and the size of paper available.

Example 2. Plot a graph from the following table showing the relation between the radius of a circle in inches and its area in square feet:

Radius in inches .	5	10	15	20	25	30
Area in square feet .	0·5	2·2	4·9	8·7	13·6	19·6

From this graph find the area of a circle of radius two feet, and the radius of the circle which has an area of 10 square feet. The scales should be chosen so as to obtain a figure of convenient size. Assuming that the available squared paper is $3'' \times 4''$, we mark the radius on the horizontal axis OX with a scale $1''$ to represent a radius of $10''$, thus permitting the radius of $30''$ to be included. The area we mark on the vertical axis OY with a scale $1''$ to represent an area of 5 sq. ft., thus permitting the area of 19·6 sq. ft. to be included. The variable to be represented along each axis should be written down, and the divisions along the axis should be numbered in accordance with the scale chosen.

The points A, B, C, D, E, F are plotted in accordance with the table, and are then joined to form a smooth curve.

The point P on this graph represents a radius of two feet (24 inches), and the length LP represents the area of this circle. It is seen to be 12·6 sq. ft.

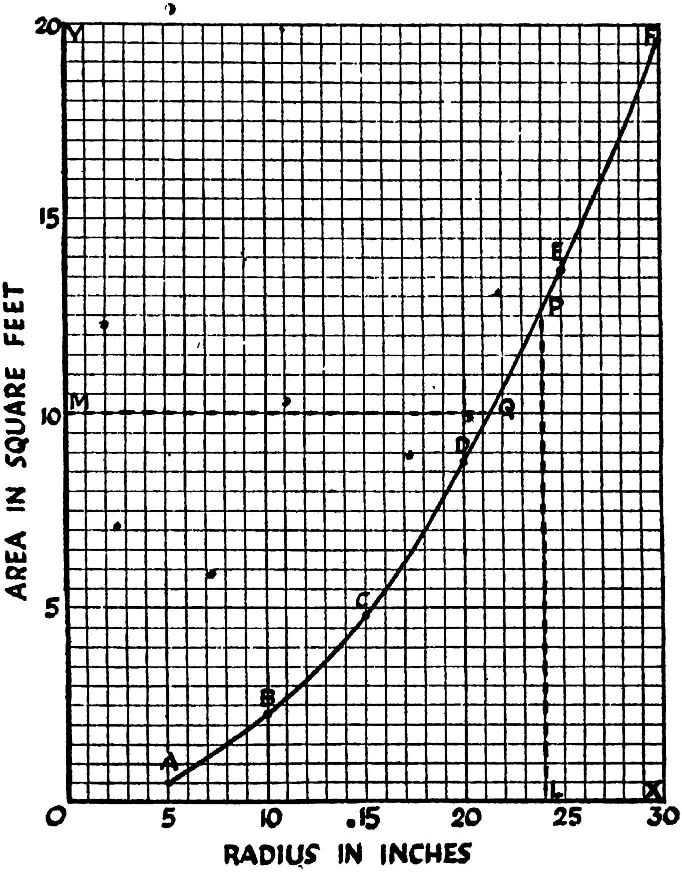

FIG. 14

The point Q represents an area of 10 sq. ft., and the length MQ represents the radius of this circle. It is seen to be 21·4 ins.

The questions in Exercises 13b and 13c, pages 189-193, should now be attempted.

Travel Graphs

71. Example 3. A man leaves a place three miles due north of his home at 2 o'clock, and walks northwards at the rate of 4 m.p.h. for $2\frac{1}{2}$ hours. He then rests an hour for tea, after which he resumes his walk northwards for a further three hours at the rate of 3 m.p.h. Find how far he was from his home at 6 o'clock, and the time at which he was 20 miles from home.

In Fig. 15, lengths measured in a direction parallel to OX represent the time after 2 o'clock to a scale of $\frac{1}{2}$ inch to an hour.

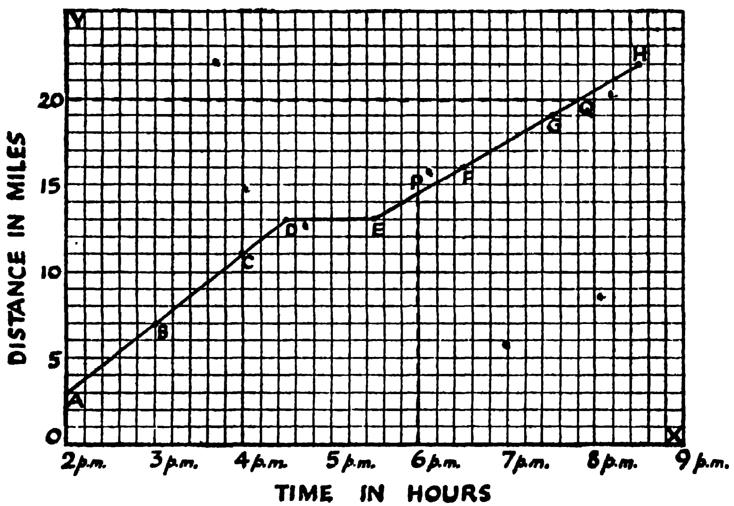

FIG. 15

Lengths measured in a direction parallel to OY represent the distances from home to a scale of $\frac{1}{10}$ inch to a mile. The point A records the fact that he is 3 miles from home at 2 o'clock. As he walks at 4 m.p.h., his distance from home at 3 o'clock is 7 miles. This is recorded by the point B. Similarly C and D record distances of 11 miles and 13 miles at 4 o'clock and 4.30 respectively. He rests an hour, and thus at 5.30 is still 13 miles from home, as recorded by the point E. He now walks at 3 m.p.h., and the points F, G, H record distances of 16, 19, 22 miles at

6.30, 7.30 and 8.30 respectively. Thus the graph A to H shows the man's position at any time between 2 o'clock and 8.30.

The point P records the man's position at 6 o'clock. It is seen that he is then $14\frac{1}{2}$ miles from home.

The point Q records the time at which the man is 20 miles from home. It is seen that the time is 7.50.

Note.—On reference to the graph it will be observed that *while the man travels at a uniform rate, his graph is a straight line.* Since two points determine a straight line, it will be sufficient to plot two points only for each portion of the graph corresponding to a uniform rate of walking. Thus the graph just discussed could have been obtained by plotting the points A, D, E, H only.

Example 4. A man leaves home at 1 o'clock, and cycles at a steady rate of 9 m.p.h. for $2\frac{1}{2}$ hours. He then rests for $\frac{1}{4}$ of an hour, and continues at the rate of 15 miles an hour till 6.15. He then rests for $\frac{1}{2}$ an hour, and cycles back at a uniform rate, arriving at his home at 11.45. At what rate did he cycle home ?

In Fig. 16 lengths measured in a direction parallel to OX represent the time after 1 o'clock to a scale of $\frac{1}{8}$ inch to an hour. Lengths measured in a direction parallel to OY represent distances from home to a scale of $\frac{1}{20}$ of an inch to a mile.

The point O records the fact that he leaves home at 1 o'clock. Two and a half hours later he will have cycled $2\frac{1}{2}\times 9=22\frac{1}{2}$ miles. The point A records the fact that at 3.30 he is $22\frac{1}{2}$ miles from home. The point B records the fact that at 3.45 he is still $22\frac{1}{2}$ miles from home. He now cycles at 15 m.p.h. The point C records the fact that one hour later, that is at 4.45, he is at a distance of $22\frac{1}{2}+15=37\frac{1}{2}$ miles from home. He continues to cycle at this uniform rate till 6.15, so that the straight line BC is produced to D which records his position at 6.15. The point E records the fact that at 6.45 he is still the same distance from home as at 6.15. The point F records the fact that he arrives back home at 11.45.

As we are told that he cycles home at a uniform rate, the journey home is represented by the straight line EF. To obtain this rate, we have to determine how far he travels in an hour. If we refer to the graph, we observe that E records the fact that he is 60 miles from home at 6.45. The point P records that one

hour later, that is at 7.45, he is 48 miles from home. Thus he has travelled 12 miles in the hour.

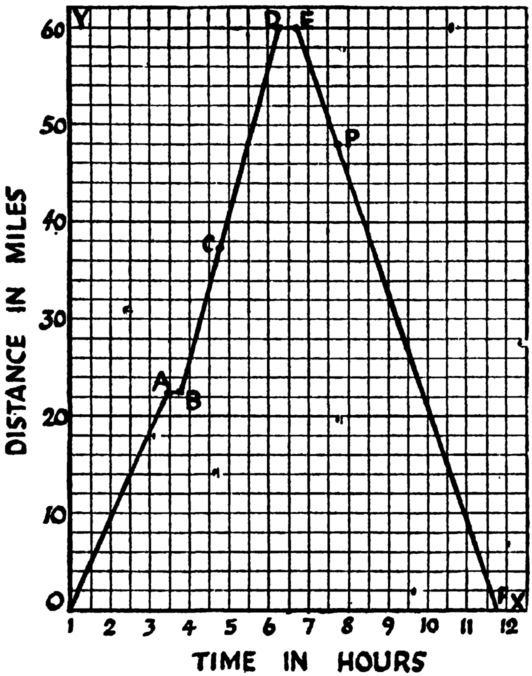

FIG. 16

Note.—In choosing two points on the line EF to find his rate, greater accuracy is assured if the points are taken as far apart as possible. Thus taking E and F as the two points, we find that he cycles 60 miles in 5 hours, which is at a rate of 12 miles per hour.

Intersection of Two Graphs

72. Example 5. Two towns A and B are 450 miles apart. An aeroplane X starts from A at 3 o'clock, and travelling at a uniform rate arrives at B at 8 o'clock. Another aeroplane Y leaves B at 4 o'clock, and travelling uniformly arrives at A at 7.45. Find (a) when and where the aeroplanes crossed ; (b) how

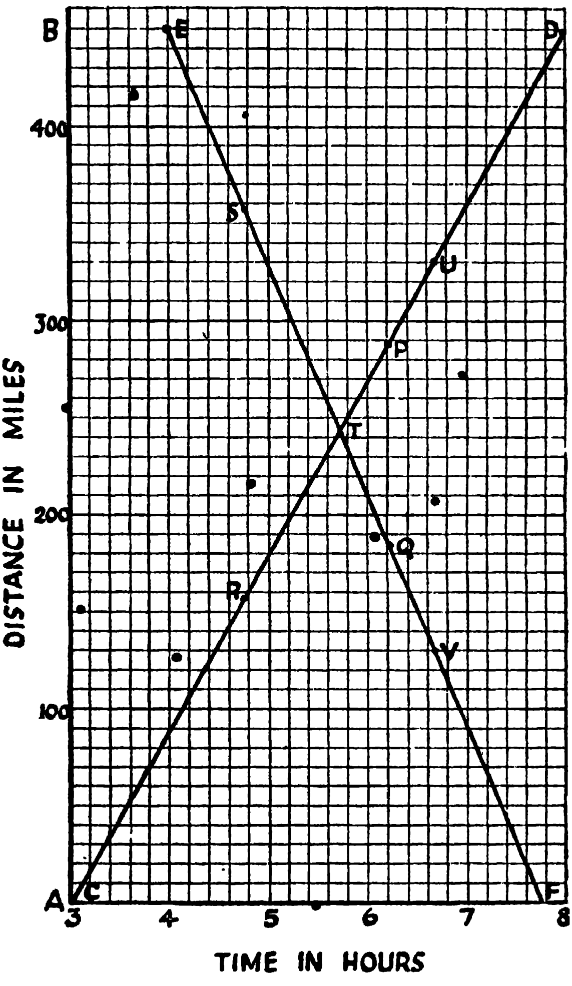

FIG. 17

far apart they were half an hour after crossing; (*c*) the times at which they were 200 miles apart.

In Fig. 17 horizontal lengths represent times after 3 o'clock to a scale of half an inch to the hour, and the vertical lengths

represent distances to a scale of 1 inch to 100 miles. C and D record that X is at A at 3 o'clock and at B at 8 o'clock; so that CD is the graph of X's progress.

E and F record that Y is at B at 4 o'clock and at A at 7.45; so that EF is the graph of Y's progress. Now at the instant that X and Y cross, they are at the same place at the same time; in other words, the point recording that place and time must be on each graph. Thus *their meeting-place and time are given by the point of intersection of the graphs*, that is by the point T.

(*a*) It will be seen that they meet about 244 miles from A, at 5.43 approximately.

(*b*) Half an hour after the aeroplanes meet, that is at 6.13, the positions of X and Y are recorded by the points P and Q respectively. It is seen that their distance apart is 105 miles.

(*c*) To find the time at which the aeroplanes are 200 miles apart, we have to discover the time at which a vertical line cuts the graph in two points whose distance apart represents 200 miles. This can be obtained by marking two points on a piece of paper to represent 200 miles, that is two points whose distance apart is 2 inches. Now slide the paper in a direction parallel to AB with one point on one of the graphs, and observe the position at which the other point lies on the other graph. It will be found that there are two such positions: (1) when the points are at R and S; (2) when they are at U and V. These correspond to the times 4.46 and 6.40 respectively.

The questions in Exercise 13d, page 193, should now be attempted.

Positive and Negative Values of the Variable

73. In the graphs considered so far, the variables had positive values only. Thus it was not intelligible to consider a race $-\frac{1}{2}$ mile in length, or a circle of area -10 sq. ft. In some cases, however, it may be necessary to consider negative values of the variables. For this purpose the convention illustrated in Fig. 18 is employed.

Suppose that O is the origin, and OX, OY are the axes as defined in § 68. Suppose also that the variable measured along the direction OX is called 'x,' and the connected variable measured along the direction OY is called 'y.' Then plotting the point A records the fact that when x has the value 3, y has the value 2.

To plot negative values, we adopt the convention that length measured in a direction opposite to that used for positive values of the variable, is to be considered as negative (see § 13). Thus lengths measured in the direction OX' represent negative values of x. Hence plotting the point B records the fact that when x has the value -2, y has the value 4.

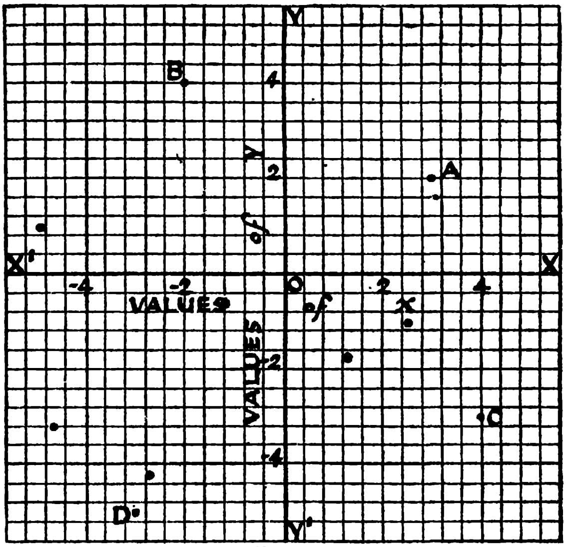

FIG. 18

In the same way lengths measured in the direction OY' represent negative values of y. Thus plotting the point C records the fact that when x has the value 4, y has the value -3. It is seen that the point D indicates that $x=-3$, $y=-5$.

$X'OX$ may be spoken of as the x-axis, and $Y'OY$ as the y-axis.

It is important to notice that **for every point on the x-axis the value of y is 0,** and **for every point on the y-axis the value of x is 0.**

Graphs of Algebraic Functions

74. Hitherto the table of values required to construct a graph has been supplied. In other words, the graph has been one of statistics. It may, however, be necessary to construct our own table of values before the graph can be drawn.

Example 6. Plot the graph, between the values $x=-3$ and $x=+3$, showing the relation between y and x, where these two variables are connected by the relation $y=2x+2$.

When x has the value -3, y assumes the value $2(-3)+2=-4$. Thus corresponding values of x and y may be calculated as shown in the vertical columns of the following table:

When $x=$	-3	-2	-1	0	1	2	3
$2x=$	-6	-4	-2	0	2	4	6
$y=2x+2=$	-4	-2	0	2	4	6	8

In Fig. 19, $X'OX$ represents the x-axis, and $Y'OY$ the y-axis. The points given by the pairs of values in the table are plotted at A, B, C, D, E, F, G. It is seen that they all lie on a straight line.

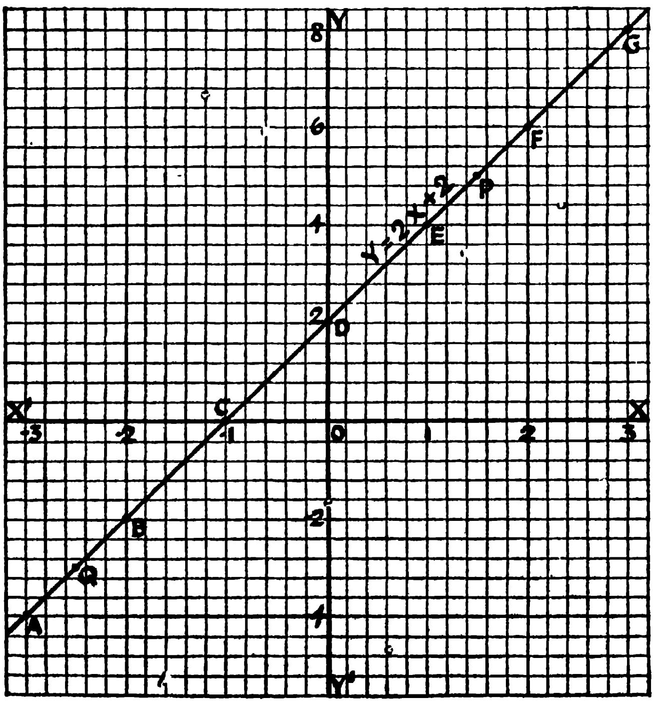

FIG. 19

In the example, we were asked to plot the graph between the values $x=-3$ and $x=+3$. Had the graph been plotted for values of x less than -3 and greater than $+3$, it would have been found that all the points obtained were on the straight line shown in Fig. 19 *when produced in both directions*. This line, so produced, is the graph of the relation $y=2x+2$, and may be called for short the graph of the expression $2x+2$.

There are two important points which should be noted.

1. **Any pair of corresponding values of x and y which satisfy the relation $y=2x+2$ gives a point on the graph.** Thus when $x=1\frac{1}{2}$, then $y=2(1\frac{1}{2})+2=5$. It is seen that this point is on the graph, namely at P.

2. **The co-ordinates of any point on the graph satisfy the relation $y=2x+2$.** Thus the co-ordinates of the point Q on the graph are $x=-2\frac{1}{2}$, $y=-3$. It is seen that $-3=2(-2\frac{1}{2})+2$, so that the relation $y=2x+2$ is satisfied.

The complete graph of the relation $y=2x+2$ is a straight line of infinite length, so that a portion of it only can be plotted. The limits of the values of x between which the graph is plotted is called **the range** of x. Corresponding values of the variables such as $x=-3$, $y=-4$ and $x=0$, $y=2$ are usually shown in the abbreviated form $(-3, -4)$ and $(0, 2)$. The first number in the brackets is the value of x, that is of the independent variable, and the second number is the value of y, that is of the dependent variable or of the expression.

75. We have seen that the expression $2x+2$ assumes different values as the value of x changes, but that for any chosen value of x the expression $2x+2$ has a definite value. When a change is produced in one variable by a change in another, we say that the first or dependent variable is a **function** of the second or independent variable. (See § 69.) Thus the area of a circle is a function of its radius; the velocity of a falling stone is a function of the distance it has fallen; the cost of a motor-car is a function of its horse-power.

It must be understood that when we state that one thing is a function of another, it is not essential that an algebraic expression exists, by means of which values of the function can

be calculated. Thus there is no expression by which the cost of a motor-car can be calculated from its horse-power. When there is such an expression, as in the case of the area of a circle in terms of its radius, we may call it an **algebraic function.**

Thus $2x+2$ is an algebraic function of x. The graph of Fig. 19 may be described as one showing the variation of the function $2x+2$ with the independent variable x. The symbols $f(x)$ and $F(x)$ are frequently used to indicate functions of the independent variable x.

Example 7. Plot a graph to show the change in the function $f(x)$ for values of x from -3 to $+3$, when $f(x)=3x-1$. The table of values is:

$x=$	-3	-2	-1	0	1	2	3
$3x=$	-9	-6	-3	0	3	6	9
$f(x)=3x-1=$	-10	-7	-4	-1	2	5	8

The graph is shown in Fig. 20.

The questions in Exercise 13e, page 195, should now be attempted.

76. Example 8. Plot the graphs of the relations (1) $y=4x$, (2) $y=4x+3$, (3) $y=4x-5$, for the range $x=-5$ to $x=5$, all with the same axes and scales.

The tables of values are:

(1)

$x=$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y=4x=$	-20	-16	-12	-8	-4	0	4	8	12	16	20

(2)

$x=$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$4x=$	-20	-16	-12	-8	-4	0	4	8	12	16	20
$y=4x+3=$	-17	-13	-9	-5	-1	3	7	11	15	19	23

(3)

$x=$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$4x=$	-20	-16	-12	-8	-4	0	4	8	12	16	20
$y=4x-5=$	-25	-21	-17	-13	-9	-5	-1	3	7	11	15

From these three tables of values it will be seen that x is to have a range from -5 to $+5$, that is of 10, whereas y is to have a range from -25 in table (3) to $+23$ in table (2), that is a range

of 48. Now if we use the same scale for y as for x, we should require a paper about five times as long as it is wide. It is

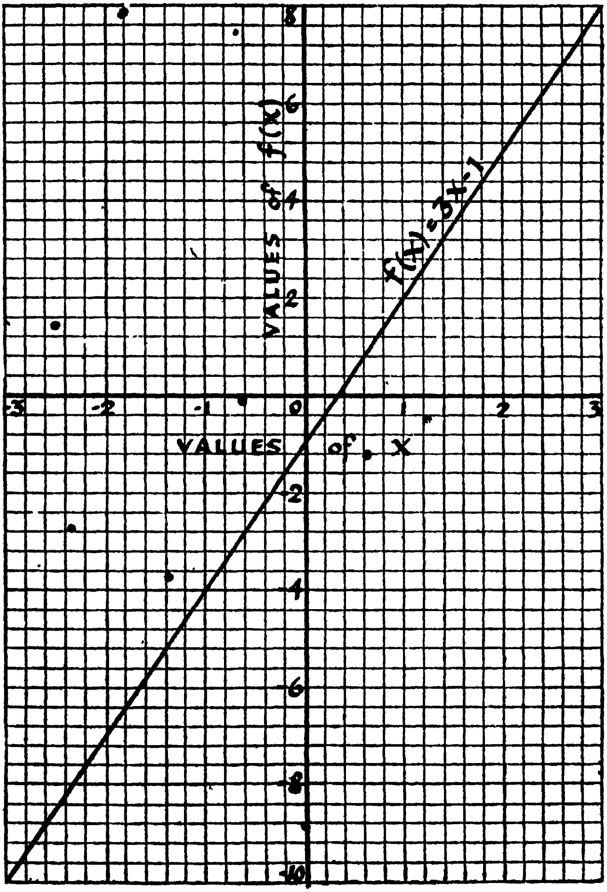

FIG. 20

better, therefore, to use a smaller scale for y than that used for x. The pupil should always consult the extreme values in his complete table to decide what scales he should adopt for x and y.

Fig. 21 shows the three graphs drawn to a scale:

1 inch represents 4 for x and 1 inch represents 20 for y.

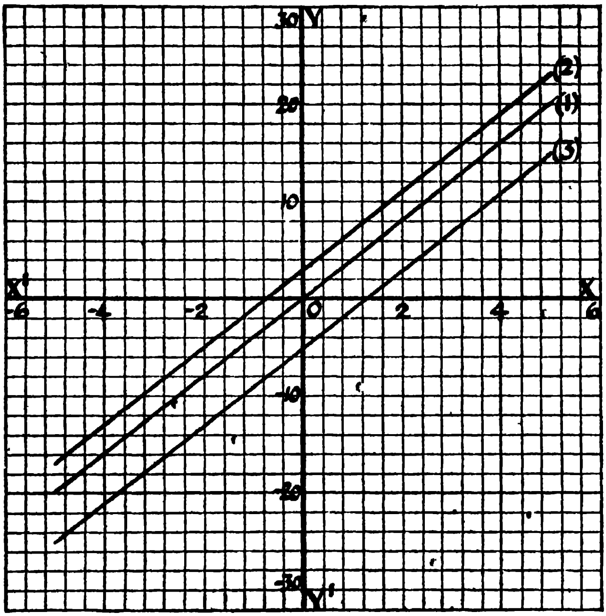

FIG. 21

Condition for Parallel Lines

77. It is seen that for any chosen value of x, the value of y in $y=4x+3$ is greater by 3 than the value of y in $y=4x$. Thus the ordinate in (2) in Fig. 21 is greater by 3 than the ordinate in (1), for any chosen value of x. It follows that the graph of (2) must be parallel to the graph of (1).

In the same way, for any selected value of x, the ordinate in (3) is 5 less than the ordinate in (1), so that the graphs of (3) and (1) are also parallel.

We infer that **if a has a fixed value but b varies in the relation $y=ax+b$, the graphs obtained will be parallel straight lines.**

One of these parallel lines will pass through the origin. For this line the values $x=0$, $y=0$ must satisfy the relation $y=ax+b$.

$$\therefore \quad 0=a\times 0+b. \qquad \therefore \quad b=0.$$

Thus the line parallel to $y=ax+b$ which passes through the origin is $y=ax$.

Intercepts on the Axes

78. Where a graph cuts the x-axis, y must be 0. In the relation $y=4x-5$, if $y=0$, $x=\frac{5}{4}$. Thus the graph of $y=4x-5$ cuts the x-axis at the point $(\frac{5}{4}, 0)$. See (3) in Fig. 21.

Where a graph cuts the y-axis, x must be 0. In the relation $y=4x-5$, if $x=0$, $y=-5$. Thus the graph of $y=4x-5$ cuts the y-axis at the point $(0, -5)$. See (3) in Fig. 21.

The distances from the origin to the points where a straight line cuts the axes are called the **intercepts** on the axes.

79. Example 9. Plot the graphs of the relations

(1) $y=3+x$, (2) $y=3+3x$, (3) $y=3-2x$,

for the range $x=-4$ to $x=5$.

The tables of values are:

(1) $x=$	-4	-3	-2	-1	0	1	2	3	4	5
$y=3+x=$	-1	0	1	2	3	4	5	6	7	8

(2) $x=$	-4	-3	-2	-1	0	1	2	3	4	5
$y=3+3x=$	-9	-6	-3	0	3	6	9	12	15	18

(3) $x=$	-4	-3	-2	-1	0	1	2	3	4	5
$y=3-2x=$	11	9	7	5	3	1	-1	-3	-5	-7

Fig. 22 shows the three graphs.

The Meaning of b in $y=ax+b$

80. The graph of $y=ax+b$ cuts the y-axis where $x=0$. If $x=0$, then $y=b$ for all values of a, so that the graph cuts the y-axis at the point $(0, b)$. Thus **the graph of $y=ax+b$ makes an intercept b on the y-axis.**

It is seen that the constant term in (1), (2) and (3) in Example 9 is in each case 3.

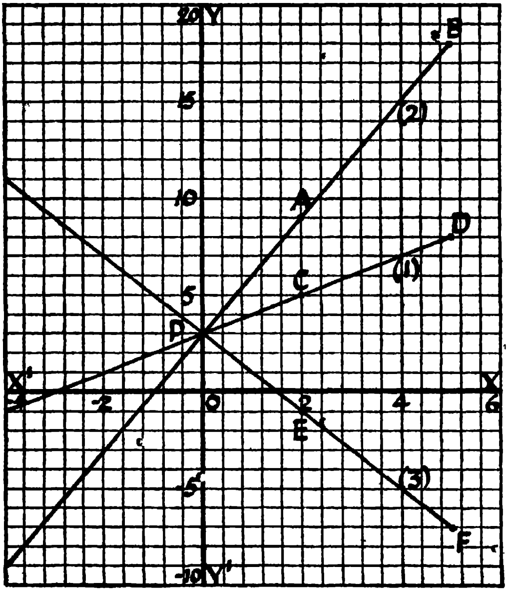

FIG. 22

Hence each graph cuts the y axis at the same point, namely (0, 3). This is the point marked P in Fig. 22.

Gradients

The Meaning of a in $y=ax+b$

81. We use the word slope or **gradient** of a straight line to indicate the ratio of the increase in the value of y to the increase in the value of x between any two points on the line.

Consider the points P and A on (2), that is the points (0, 3) and (2, 9). The value of y increases by 6 and that of x by 2. Hence the ratio of these increases is $\frac{6}{2}$, that is 3. Any other pair of points on this line give the same ratio. Thus the points A and B, that is the points (2, 9) and (5, 18), give increases of 9 for

y and 3 for x. Hence the ratio is again 3. Thus the gradient of line (2) is 3.

Note that the coefficient of x in (2) is 3.

Consider next the points P and C on (1), that is the point (0, 3) and (2, 5). These give a ratio $\frac{2}{2}=1$. In the same way the points C and D, that is the points (2, 5) and (5, 8), give a ratio $\frac{3}{3}=1$. Thus the gradient of line (1) is 1.

Note that the coefficient of x in (1) is 1.

In line (3) the points P and E, that is the points (0, 3) and (2, −1), give a ratio $\frac{-4}{2}=-2$. In the same way the points E and F, that is the points (2, −1) and (5, −7), give a ratio $\frac{-6}{3}=-2$. Thus the gradient of line (3) is −2.

Note that the coefficient of x in (3) is −2.

We infer that **the gradient of the graph obtained from the relation $y=ax+b$ is a.**

The General Linear Relation

82. Consider the general relation $ax+by+c=0$, where x and y are the variables, and a, b and c are numbers.

This relation can be written :

$$by=-ax-c,$$

whence

$$y=-\frac{a}{b}x-\frac{c}{b} \quad . \quad . \quad . \quad . \quad (1)$$

By comparing (1) with the form hitherto used, namely $y=ax+b$, we see that the graph of (1), and therefore **the graph of $ax+by+c=0$ is a straight line whose intercept on the y-axis is $-\frac{c}{b}$, and whose gradient is $-\frac{a}{b}$.** (See §§ 80, 81.)

We therefore call an equation of the form $ax+by+c=0$, or $y=mx+n$, a **linear equation,** and an expression of the form $ax+b$, **a linear function of x.**

83. A straight line is fixed when two points on it are known. Hence to obtain the graph of the relation $ax+by+c=0$, it is sufficient to determine two pairs of corresponding values of x and y, and join the points they indicate by a straight line.

Example 10. Find the equation of the straight line which passes through the points (1, 6) and (4, 18).

The ratio of the increase of y to that of x is $\frac{12}{3}=4$.

Hence the equation of the graph must be of the form $y=4x+b$. (See § 81.) Since the graph passes through the point (1, 6), we must have $6=4\times1+b$; whence $b=2$.

Thus the required equation is $y=4x+2$.

Alternatively we may proceed as follows:

The equation of any straight line is of the form $y=ax+b$. If this line is to pass through (1, 6) and (4, 18), we must have $6=a+b$ and $18=4a+b$. These equations when solved give $a=4$, $b=2$. Thus the required equation is $y=4x+2$.

Fig. 23 shows the graph of $y=4x+2$ obtained from the table of values:

$x=$	-1	3	5
$y=4x+2=$	-2	14	22

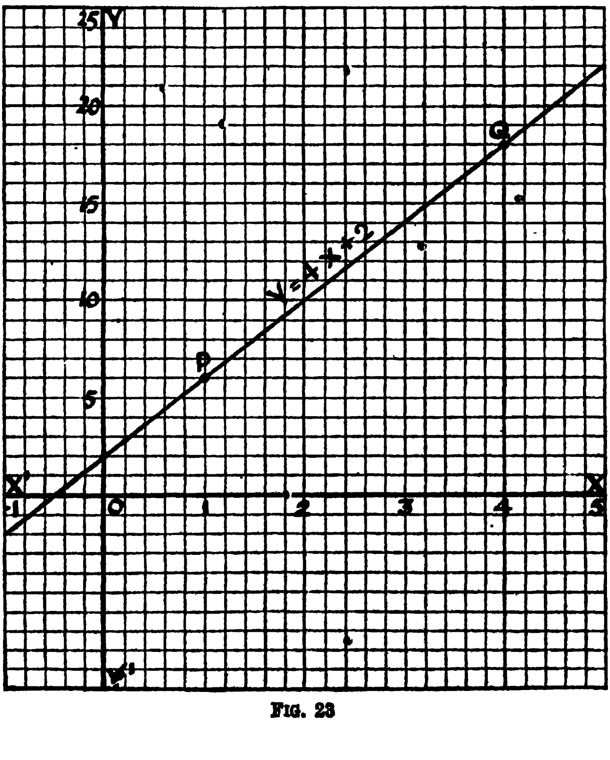

FIG. 23

It is seen that the graph passes through the points (1, 6) and (4, 18), marked P and Q respectively.

The questions in Exercise 13f, page 197, should now be attempted.

Solution of Simultaneous Equations by Graphs

84. Example 11. Draw graphs of the relations (1) $y=2x-3$; (2) $4y-x=2$, and then solve the simultaneous equations.

The second relation can be put in the form $y=\frac{x+2}{4}$.

Using the table of values:

(1)

$x=$	-2	6
$y=2x-3=$	-7	9

(2)

$x=$	-2	6
$y=\frac{x+2}{4}=$	0	2

we obtain Fig. 24.

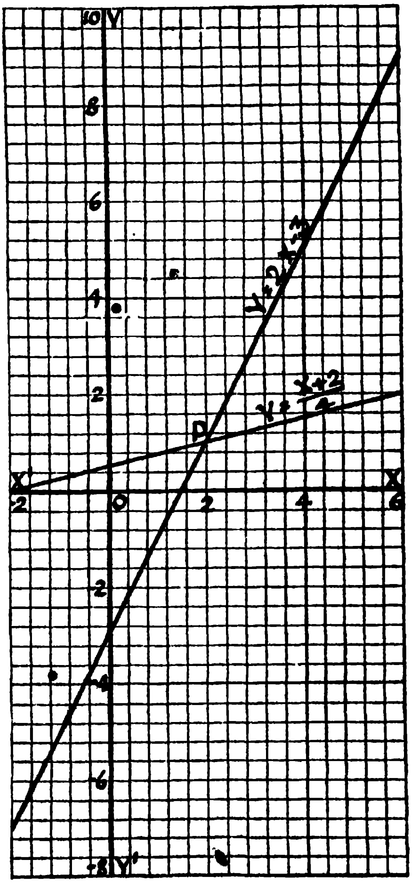

Fig. 24

It is seen that the two graphs intersect at the point P, whose co-ordinates are $x=2$, $y=1$. Now the co-ordinates of any point on (1) satisfy the equation $y=2x-3$, and the co-ordinates of any point on (2) satisfy the equation $4y-x=2$. But the point P lies on both (1) and (2). Hence its co-ordinates satisfy both equations. Thus the solution of the simultaneous equations is $x=2$, $y=1$.

We infer that **the solution of two simultaneous equations may be obtained**

by plotting the graphs corresponding to the two equations, and reading off the co-ordinates of the point or points of intersection.

85. It has been shown that the graph of a linear relation connecting x and y is a straight line, and two straight lines can intersect in one point only; it therefore follows that there can be only one solution to a pair of linear simultaneous equations.

A pair of relations such as $3y=2x+5$ and $6y=4x+3$ can be put into the forms $y=\frac{2}{3}x+\frac{5}{3}$ and $y=\frac{2}{3}x+\frac{1}{2}$. It follows from § 77 that their graphs are parallel lines, and so will not intersect. Hence there is no solution of this pair of simultaneous equations. Such equations are said to be inconsistent.

The relations $2y=5x-6$ and $6y=15x-18$ can both be put into the form $y=\frac{5}{2}x-3$. It follows that their graphs are coincident lines, which thus have an infinite number of common points. Hence there are an infinite number of solutions of this pair of simultaneous equations. Such equations are said to be not independent.

The pupil should compare the results in this paragraph with those established in § 58.

86. Example 12. Find the equation of a straight line passing through the point of intersection of the graphs of (1) $y=4x-7$ and (2) $y=8-x$, and which is parallel to (3) $y=2x+1$.

As all the graphs are straight lines it is sufficient to plot two points for each. Tables of values for graphs (1), (2) and (3) are:

(1)			(2)			(3)		
$x=$	1	4	$x=$	1	4	$x=$	0	3
$y=4x-7=$	-3	9	$y=8-x=$	7	4	$y=2x+1=$	1	7

These three graphs are shown in Fig. 25. It will be seen that graphs (1) and (2) intersect at the point P (3, 5). The line through P parallel to (3) is marked on the diagram as (4). As (4) is parallel to the graph of $y=2x+1$, its equation must be of the form $y=2x+b$ (see § 77). Further, it is seen in Fig. 25 that the line (4) cuts the y axis where $y=-1$. Hence the equation of (4) must be $y=2x-1$. (See § 80.)

Alternatively we may proceed as follows:

The co-ordinates of the point of intersection of (1) and (2) are obtained by solving the simultaneous equations $y=4x-7$ and $y=8-x$. These give $x=3$, $y=5$.

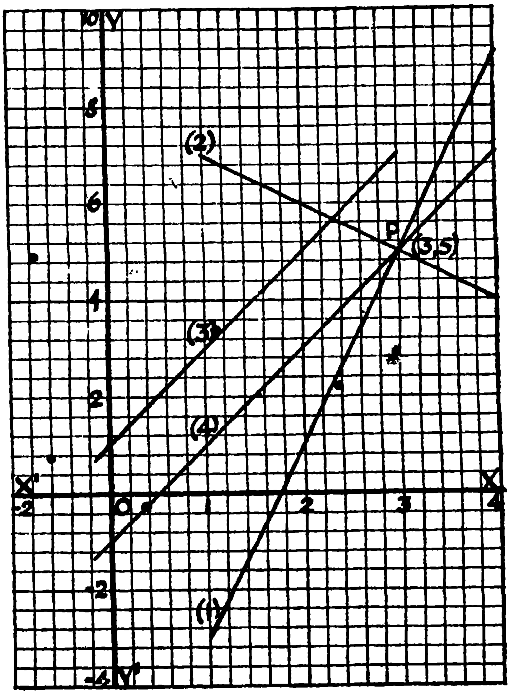

FIG. 25

The equation of any straight line parallel to $y=2x+1$ is of the form $y=2x+b$. If this line passes through the point (3, 5), we require $5=2\times3+b$. Hence $b=-1$.

Thus the required equation is $y=2x-1$.

The questions in Exercise 13g, page 200, should now be attempted.

Exercise 13a

The marriage rates per 1000 for Gt. Britain and N. Ireland were as follows:

Year	1924	1925	1926	1927	1928	1929	1930
Rate	15·0	14·9	14·1	15·3	15·1	15·5	15·5

Year	1931	1932	1933	1934	1935	1936
Rate	15·3	15·0	15·5	16·6	16·8	17·0

Represent these data on a diagram and use it to find:

1. the year when the rate was at a maximum;
2. the year when the rate was at a minimum;
3. the period of 12 months when there was least change in the rate;
4. the period of 12 months when there was most change in the rate.

The number of elementary classes in England and Wales having not more than 20 pupils were as follows:

Year	1930	1931	1932	1933	1934	1935	1936
Classes	13,900	14,200	12,800	11,600	11,700	12,500	13,600

Represent these data on a diagram and use it to find:

5. the year when the number of classes was least;
6. the period of 12 months when there was the greatest rise in the number of classes;
7. the period of 12 months when there was the greatest drop in the number of classes.

A boy is given a fixed weekly sum for pocket-money and saves the following amounts:

Week	1	2	3	4	5	6	7	8
Saving in pence	6	$4\frac{1}{2}$	7	$9\frac{1}{2}$	0	5	8	$3\frac{1}{2}$

Represent these data on a diagram and use it to find:

8. the week when he saved most;
9. the week when he spent most;
10. the number of weeks in which he saved more than sixpence.

The mean barometer readings at Greenwich from Jan. 11th to Jan. 20th are given in inches, to the nearest $\frac{1}{100}$th of an inch, for the years 1936 and 1937:

	Jan. 11	Jan. 12	Jan. 13	Jan. 14	Jan. 15
1936 .	29·98	30·06	30·04	30·14	29·94
1937 .	30·10	29·97	29·91	29·97	29·80

	Jan. 16	Jan. 17	Jan. 18	Jan. 19	Jan. 20
1936 .	29·39	29·24	29·24	29·16	28·81
1937 .	29·56	29·45	28·97	29·26	29·55

The diagram is shown in Fig. 26. Use this diagram to find, within the given period:

11. the date of maximum reading in 1936;
12. the date of maximum reading in 1937;
13. the date of minimum reading in 1936;
14. the date of minimum reading in 1937;
15. the date when the 1936 reading was the maximum above the 1937 reading;
16. the date when the readings for the two years showed the maximum difference;
17. the dates when the readings were increasing for both years;
18. the maximum fall during a period of 24 hours, and when this occurred;
19. the period of 24 hours for which the reading was unchanged;
20. the two consecutive days on which the readings rose at a steady rate;

The mean barometer readings at Greenwich from Jan. 21st to Jan. 30th are given in inches, to the nearest $\frac{1}{100}$th of an inch, for the years 1936 and 1937:

	Jan. 21	Jan. 22	Jan. 23	Jan. 24	Jan. 25
1936	29·07	29·34	29·61	29·40	29·15
1937	29·40	29·58	29·57	29·12	29·20

	Jan. 26	Jan. 27	Jan. 28	Jan. 29	Jan. 30
1936	29·25	29·23	29·20	28·94	29·34
1937	29·40	29·10	28·94	29·10	29·11

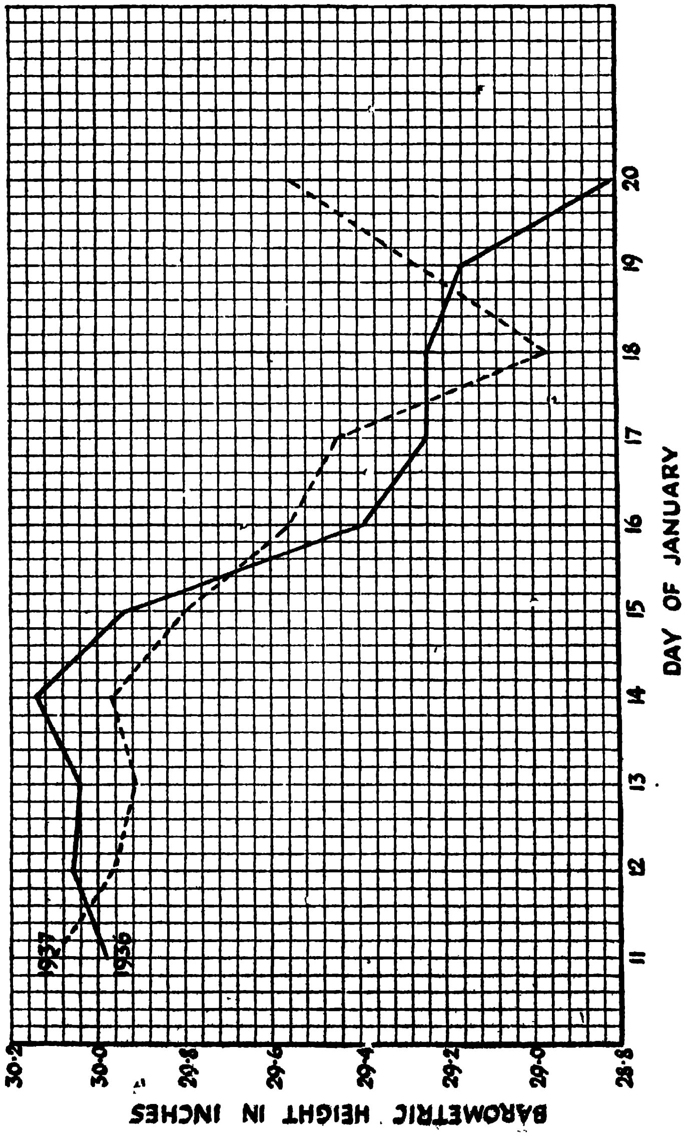
30·2
30·0
29·8
29·6
29·4
29·2
29·0
28·8
BAROMETRIC HEIGHT IN INCHES
11
12
13
14
15
16
17
18
19
20
DAY OF JANUARY
1937
1936

FIG. 26

Represent these data on a diagram and use it to find, within the given period :

21. the date of maximum reading in 1936 ;
22. the date of maximum reading in 1937 ;
23. the date of minimum reading in 1936 ;
24. the date of minimum reading in 1937 ;
25. the date when the 1936 reading was the maximum amount above the 1937 reading ;
26. the date when the readings for the two years showed the maximum difference ;
27. the maximum rise during a period of 24 hours and when it occurred ;
28. the maximum fall during a period of 24 hours and when it occurred ;
29. the period of 24 hours during which the reading rose by not more than 0·01 inch ;
30. the period of 24 hours during which the reading fell by not more than 0·01 inch.

The monthly rainfalls at Greenwich from January to September are given in inches, to the nearest $\frac{1}{20}$th of an inch, for the years 1936 and 1937 :

	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.
1936	3·55	1·45	0·80	1·70	0·40	3·35	2·95	0·55	3·15
1937	3·60	3·95	2·90	2·45	3·35	1·85	0·70	1·70	1·25

The diagram is shown in Fig. 27. Use this diagram to find, within the given period :

31. which was the wettest month in 1936 ;
32. which was the wettest month in 1937 ;
33. which was the driest month in 1936 ;
34. which was the driest month in 1937 ;
35. which month had the maximum amount more rain in 1936 than 1937 ;
36. which month was the maximum amount drier in 1936 than 1937 ;
37. which was the wettest month (taking the two years into consideration) ;
38. which was the driest month (taking the two years into consideration) ;
39. in which month the two rainfalls were most nearly equal ;
40. which year had most months with a rainfall greater than 2 inches.

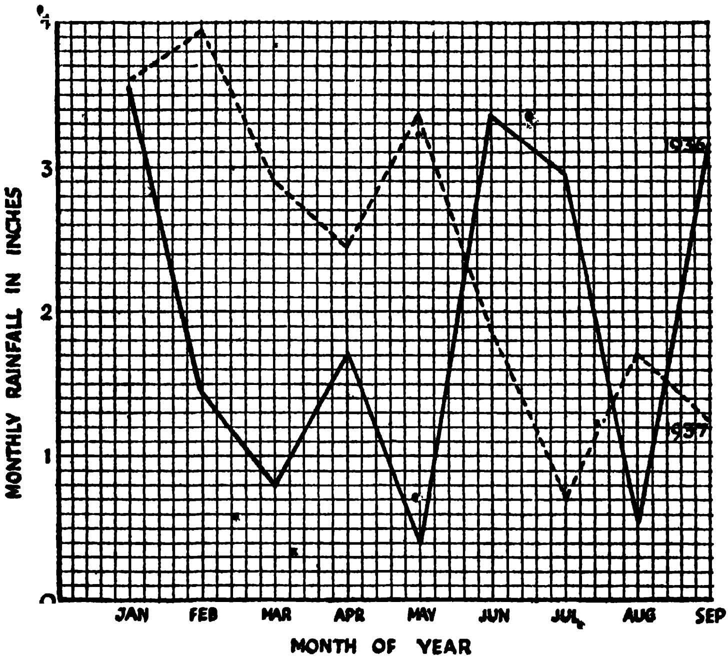

FIG. 27

The monthly rainfalls at Greenwich from January to September are given in inches, to the nearest $\frac{1}{20}$th of an inch, for the years 1934 and 1935:

	Jan.	Feb.	Mar.	April	May	June	July	Aug.	Sept.
1934	1·40	0·15	2·15	2·15	0·40	1·40	0·90	1·70	1·20
1935	1·25	2·15	0·60	2·90	1·55	2·85	0·55	2·20	2·90

Represent these data on a diagram and use it to find, within the given period:

41. which were the wettest months in 1934;

42. which were the wettest months in 1935;

43. which was the driest month in 1934;

44. which was the driest month in 1935;

45. which month had the maximum amount more rain in 1934 than 1935;

46. which month was the maximum amount drier in 1934 than 1935;

47. which was the wettest month (taking the two years into consideration);

48. which was the driest month (taking the two years into consideration);

49. in which month the rainfalls were most nearly equal;

50. which year had most months with a rainfall greater than one inch.

The birth and death rates per 1000 for Gt. Britain and N. Ireland were as follows:

	1924	1925	1926	1927	1928	1929	1930
Births .	19·3	18·7	18·2	17·1	17·2	16·7	16·8
Deaths	12·6	12·4	11·9	12·5	11·9	13·6	11·7

	1931	1932	1933	1934	1935	1936
Births .	16·3	15·8	14·9	15·2	15·2	15·3
Deaths	12·5	12·3	12·5	12·0	12·0	12·3

Represent these data on a diagram and use it to find:

51. in what year the birth rate was at a minimum;

52. in what year the death rate was at a minimum;

53. in what year the death rate was at a maximum;

54. in what year the population per 1000 increased (*a*) most; (*b*) least.

The maximum and minimum temperatures at Greenwich for the first 12 days of January 1937 were, in degrees Fahrenheit:

	Jan. 1	Jan. 2	Jan. 3	Jan. 4	Jan. 5	Jan. 6
Max.	50·5	50·8	53·0	49·6	44·5	53·5
Min.	40·0	37·6	47·9	39·8	34·3	42·3

	Jan. 7	Jan. 8	Jan. 9	Jan. 10	Jan. 11	Jan. 12
Max.	50·0	42·1	43·6	42·9	47·0	48·2
Min.	39·9	30·0	30·7	29·5	29·8	32·7

Represent these data on a diagram and use it to find :

55. on which date the highest temperature was recorded;
56. on which date the lowest temperature was recorded;
57. on which date there was the greatest temperature variation, and the amount of this variation;
58. on which date there was the least temperature variation, and the amount of this variation ;
59. (*a*) the highest ; (*b*) the lowest average temperature. (Make a table of the daily average temperature, and draw a diagram.)

The daily hours of sunshine recorded at Greenwich from July 20th to July 28th are given for the years 1936 and 1937 :

July	20	21	22	23	24	25	26	27	28
1936	6·7	3·6	4·1	0	12·1	7·6	7·9	10·3	0·5
1937	6·3	2·0	4·3	2·6	9·0	0·7	0·5	0	1·6

Represent these data on a diagram and use it to find :

60. which dates were sunnier in 1937 than in 1936;
61. which date showed the greatest difference in hours of sunshine for the two years;
62. which date showed the least difference in hours of sunshine for the two years;
63. which two consecutive days showed (*a*) the greatest; (*b*) the least difference in hours of sunshine.

A boy's weekly marks for arithmetic, algebra and geometry for the ten weeks of the autumn term were :

Week .	1	2	3	4	5	6	7	8	9	10
Arithmetic	73	81	83	61	60	66	70	79	75	72
Algebra .	55	68	61	73	62	69	72	63	65	69
Geometry	49	43	52	68	54	59	69	74	62	66

Make a table of his weekly average arithmetic and algebra mark, and draw a diagram to represent this and his geometry mark. Use this diagram to find :

64. the weeks when his geometry mark was higher than his average arithmetic and algebra mark;

65. the week when there was (*a*) the greatest; (*b*) the least difference between these two marks;

66. the week when the average of the two marks shown on the diagram was nearest to 65.

Exercise 13b

If your squared paper is $5\frac{1}{2}$ inches broad, what scale would you choose to include values of x ranging from:

1. 0 to 10. **2.** 0 to 25. **3.** 35 to 80.
4. 470 to 700. **5.** 0·09 to 0·9.

If your squared paper is 8 inches broad, what scale would you choose to include values of x ranging from:

6. 0 to 15. **7.** 25 to 60. **8.** 1350 to 2100.
9. 6·5 to 7·3. **10.** 0·003 to 0·0065.

You are provided with squared paper of length 8 inches and breadth 6 inches. State which variable you would mark along the breadth, and the scales you would choose, if the ranges required for variables A and B are:

11. A, 0 to 12; B, 4 to 20. **12.** A, 30 to 180; B, 10 to 40.
13. A, 47 to 62·5; B, 0·3 to 0·9.
14. A, 19 to 22; B, 12 to 51.
15. A, 0·9 to 1·2; B, 1·4 to 2·15.

Exercise 13c

1. From the following table of values plot a graph showing the relation between the time for which a quantity of water is heated and the temperature produced.

Temp., ° C. .	18	26	34	42	50	58
Time, mins. .	0	2	4	6	8	10

Note.—Represent temperatures vertically to a scale of 1 inch= 10° C.: time horizontally to a scale $\frac{1}{2}$ inch=1 minute.

2. Use the graph drawn in Question 1 to find the temperature of the water after it has been heated, (*a*) for 5 minutes, (*b*) for 9 minutes.

State the time taken for the water to reach a temperature of (*c*) 46° C. and (*d*) 28° C.

3. The circumference of a circle for different diameters is given in the following table:

Circum. in inches .	11	22	33	44	55	66
Diam. in inches .	$3\frac{1}{2}$	7	$10\frac{1}{2}$	14	$17\frac{1}{2}$	21

Plot a graph showing the relation of circumference to diameter, and from your graph find (*a*) the circumference of a 12-inch wheel, (*b*) the diameter of a 50-inch wheel.

4. A pipe fills a tank with water, and the depth of water in the tank at intervals of 1 minute is as follows:

Depth in inches .	0	6	12	18	24	30
Time in mins. .	0	1	2	3	4	5

Plot a graph showing the relation between depth and time. Use the graph to find (*a*) the depth at end of $3\frac{1}{2}$ minutes, (*b*) the time when the depth is 1 ft. $1\frac{1}{2}$ ins.

5. Plot a graph showing the relation between Fahrenheit and Centigrade temperature, making use of the following table:

°F. .	32	68	104	140	176	212
°C. .	0	20	40	60	80	100

Use this graph to find (*a*) the Centigrade temperature corresponding to 86° F., 150° F.; (*b*) the Fahrenheit temperature corresponding to 15° C., 35° C.

6. Plot a graph showing the variation in the length of a spring when a weight is attached to the free end.

Length in inches .	11	12	13	14	15	16
Weight in ozs. .	2	4	6	8	10	12

(*a*) From your graph read the weight required to make the spring 14·5 inches long.

(*b*) How long will the spring be when a 5-oz. weight is added to the end?

(*c*) By producing your graph backward, find the length of the spring when no weight is attached.

7. The following table gives the cost of fuel for running a train over a certain distance at different speeds:

Cost in £ . .	1	$2\frac{1}{4}$	4	$6\frac{1}{4}$	9
Miles per hour .	20	30	40	50	60

Plot a graph showing the relation between cost and speed. Use your graph to find (*a*) the cost of running at a speed of 45 m.p.h., (*b*) the speed of the train if the cost for the journey is £3.

8. Plot a graph from the following table of values to show the height of a bridge above the level of the water in a canal and the distance from a road parallel to the canal:

Height in feet .	1	3	6	10	12	12½	12	10	6	3	1
Distance in yds. .	4	8	12	16	20	24	28	32	36	40	44

At what distances from the road is the bridge 5 ft. above the level of the water in the canal? What is the height of the bridge 18 yds. from the road?

9. Plot a graph connecting the length of a pendulum and the time of swing to and fro.

Length of pendulum in cms. .	25	50	80	120	170	225
Time of swings in seconds .	1	1·4	1·8	2·2	2·6	3

What is the length of a pendulum which swings to and fro in 2 seconds?

How long will a 200-cm. pendulum take to make a swing to and fro?

10. The distance through which a stone falls and the time taken is shown in the following table:

Distance in feet .	16	64	144	256	400
Time in seconds .	1	2	3	4	5

Plot a graph showing the relation of distance fallen to time elapsed.

Read from your graph the time taken for the stone to fall 100 feet. How far did the stone fall in the last half second? If a stone takes 3½ seconds to reach the ground when dropped from a tower, how high is the tower?

11. The population of a certain town in thousands, to the nearest 500, was as follows:

Year	1850	1860	1870	1880	1890	1900	1910	1920	1930
Pop.	189	201½	215½	232	251½	274	301½	334½	376½

Plot a graph from these data, and use it to estimate the population in the years 1875, 1905, 1915.

12. The following table gives the volume of a sphere correct to the nearest 100 cubic inches, the radius being given in inches:

Radius .	9	10	11	12	13	14	15
Volume .	3100	4200	5600	7200	9200	11500	14100

Plot a graph from these data, and use it to find the volume of spheres of radii 10·5 ins. and 13·8 ins. respectively.

Find also the radii of spheres whose volumes are 3800 and 7600 cubic inches respectively.

13. The following table gives the distance of the visible horizon in sea miles for heights of the eye above sea level in feet:

Height of eye .	10	20	30	40	50	60	70	80
Dist. of horizon	3·35	4·75	5·80	6·75	7·50	8·25	8·90	9·50

From the graph of these data find the distance of the horizon for heights of 33 and 68 feet respectively. At what height will the horizon be 5·20 and 8·05 sea miles away respectively?

14. The amount of £100 at 5 per cent. compound interest is given in the following table to the nearest £:

No. of years . .	5	10	15	20	25	30	35	40
Amount in £'s .	128	163	208	265	339	432	552	704

From the graph obtained from this data find the amount at the end of 12 years, and the number of years for the amount to reach £500.

15. A flask of water is cooling, and the temperature in degrees Centigrade is taken at intervals of 5 minutes.

Minutes from start	0	5	10	15	20	25	30
Temperature .	57·8	48·5	41·7	36·2	31·9	28·7	26·6

Plot a graph and use it to find the temperatures after 7 and 26 minutes respectively. After what times will the temperatures be 41·1 and 30·8 degrees respectively?

16. The following table gives the number of miles per gallon a certain car can do for different speeds:

Miles per hour	10	15	20	25	30	35	40	45
Miles per gallon	22·3	28·1	32·4	35·3	36·8	37·0	35·8	32·0

Plot the graph and find the number of miles per gallon at 13 and 26·5 m.p.h. respectively. There are two speeds for

which the petrol consumption is 33·8 miles per gallon; find them. What are the speeds for a consumption of 36·4 miles per gallon?

17. A manufacturer finds that the cost per article varies according to the number he makes as follows:

No. of articles	1000	2000	3000	4000	5000	6000	7000	8000
Cost per article in £'s . .	2·50	2·27	2·07	1·89	1·74	1·60	1·48	1·37

Plot the graph and find the cost per article if he makes 3700. How many must he make if each costs £1·46?

18. The time taken by a hiker to walk a mile varies with weight of his pack as follows:

Weight of pack	0 lbs.	5 lbs.	10 lbs.	15 lbs.
Time for 1 mile	15 min. 48 sec.	16 m. 3 s.	16 m. 30 s.	17 m. 15 s.

Weight of pack	20 lbs.	25 lbs.	30 lbs.	35 lbs.
Time for 1 mile	18 min. 6 sec.	19 m. 30 s.	21 m. 15 s.	23 m. 57 s.

Plot the graph and find the time for one mile with packs of 8 and 21·5 lbs. respectively. Find the weight of the pack if his times are 17 min. and 21 min. 30 sec. respectively.

Exercise 13d

1. A lorry moving due north travels at 20 miles per hour for 3 hours and then at 15 miles per hour for 4 hours.

   Draw a graph showing the progress of the lorry.

2. A cyclist starting from home at 9 a.m. travels at 12 m.p.h. for the first 2 hours and then at 8 m.p.h. for the next hour, and completes the next 10 miles in 1 hour.

   Draw a graph and use it to find: (*a*) the distance the cyclist is from home at noon, (*b*) the time when the cyclist was 38 miles from home.

3. A man starts off at 10 a.m. and walks at 3 m.p.h. for 4 hours, rests for lunch for 1 hour, and continues after lunch at 3 m.p.h. for another 3 hours. He then returns along his track by bus and reaches his starting-point at 7.30 p.m.

   Draw a graph and use it to find:

   (*a*) the man's distance from his starting-point at 4 p.m.;

   (*b*) the times of day when the man was 10 miles from his starting-point.

4. A boat travels upstream at 6 m.p.h. for 2 hours, then at 8 m.p.h. for another hour. The boat is now left to be carried down-stream by the current, and reaches its starting-point 7 hours after the time it left.

By drawing a graph find the rate of the stream.

5. By walking at 4 m.p.h. for $1\frac{1}{2}$ hours, resting for $\frac{1}{2}$ an hour, and then cycling at 8 m.p.h. a man travels a distance of 30 miles. After resting a further $\frac{1}{2}$ hour he cycles back at a steady rate, and reaches home $8\frac{1}{2}$ hours after he left.

At what time did he start for home, and at what rate did he cycle back?

6. A motorist starts a journey northward at a point 20 miles due north of his home at 9 a.m. After travelling at 24 m.p.h. for $1\frac{1}{4}$ hours, he returns home at 30 m.p.h. and rests there for $\frac{1}{4}$ hour. He then travels north again at 20 m.p.h. for 3 hours, after which he starts for home and arrives at 5 p.m.

   (*a*) At what times was he 40 miles from home?
   (*b*) For what length of time was he more than 40 miles from home?
   (*c*) What was his distance from home at 4.30 p.m.?
   (*d*) At what rate did he make his final journey home?

7. Two railway stations L and M are 150 miles apart. A train leaves L at 9 a.m. and travelling at a uniform speed arrives at M at 12 noon. Another train leaves M at 8 a.m. and arrives at L at 1 p.m. Find (*a*) the time when the trains cross, (*b*) where the trains cross, (*c*) how far they were apart at 11 a.m., (*d*) at what times they were 24 miles apart.

8. A man starts from a town A at 8 a.m. and walking at 4 m.p.h. reaches town B at 1 p.m. A cyclist leaving B at 10.30 a.m. reaches A at 1 p.m.

   Find (*a*) where and when they crossed;
   (*b*) how far they were apart at noon;
   (*c*) at what times they were 9 miles apart.

9. Two aerodromes, A and B, are 700 miles apart. An aeroplane leaves A at 2.30 p.m. and travels at 150 m.p.h. for 1 hour. It then makes a forced landing, but after $\frac{1}{2}$ hour continues at the same rate as before. A second plane leaves A at 3.30 p.m. and travels at 150 m.p.h. for the first 2 hours, and then completes the journey at 200 m.p.h.

Find the time and place when the second plane overtakes the first. How far apart were the planes at 6.30 p.m.?

10. Two railway stations, A and B, are 180 miles apart. If a goods train leaves A at noon it reaches B at 9.30 p.m. having waited for half an hour on a siding 60 miles from B to allow an express from A to pass.

If the express left A at 4 p.m. and travelled at a uniform rate, find the earliest and latest times at which it could have arrived at B. Find the speed of the express in each case.

11. A single railway track 15 miles in length begins at a distance of 40 miles from A, and thus trains are unable to cross on this stretch. A passenger train leaves A at 4 p.m. and travels at 40 m.p.h. to a station B, 70 miles from A. An express leaves A at 4.15 p.m., and enters the single line 5 minutes before the first train. At what time does the express leave A if, travelling at the same rate as before, it completes its journey on the single track 5 minutes after the first train?

Note.—First find the speed of the express.

12. Two stations, A and B, are 90 miles apart, and are served by a single line except for a double line which is 10 miles in length and begins at 40 miles from A. If a train leaves A at 9 a.m. and travels at 20 m.p.h., it enters the double track 5 minutes before a train leaving B at 9.5 a.m. leaves the double track. At what time does the train leave B if it travels at the same speed as before and enters the double track 5 minutes before the train from B leaves it?

Exercise 13e

1. Draw axes $X'OX$ and $Y'OY$ such that values of x from -3 to $+3$ can be recorded and values of y from -4 to $+4$ can be recorded.

Indicate the points:

(*a*) $x=2, y=3$; (*b*) $x=-3, y=1$;
(*c*) $x=1, y=-2$; (*d*) $x=-2, y=-3$.

Mark these points P, Q, R and S respectively.

2. In question (1) join P to S and write down the values of x and y where this line cuts the axes.

Join Q to R and write down the values of x and y for the points where this line cuts the axes.

What are the values of x and y at the point of intersection of the lines PS and QR?

3. Plot the points given by the following table :

$x=$	-3	-1	1
$y=$	-1	1	3

Join these points by a straight line. Write down the values of x and y where this line crosses the axes.

4. Plot the points given by the following table :

$x=$	-5	-2	-1	1	2	3
$y=$	$-\frac{1}{2}$	1	$1\frac{1}{2}$	$2\frac{1}{2}$	3	$3\frac{1}{2}$

Draw a straight line through these points and write down the values of x and y where this line cuts the axes.

5. Draw a straight line through the points given by :

$x=$	-2	-1	1	2	4
$y=$	5	$3\frac{1}{2}$	$\frac{1}{2}$	-1	-4

Use the graph to find : (*a*) the value of y which corresponds to $x=3$; (*b*) the value of x which corresponds to $y=0$; (*c*) the value of y when $x=0$.

6. Plot the graph between $x=-4$ and $x=+4$ showing the relation between y and x, where these two variables are connected by the relation $y=2x+1$.

   Read from your graph the value of x which makes $y=0$, and the value of y which makes $x=0$.

7. Plot the graph between the values $x=-2$ and $x=+3$ showing the relation between y and x, where these variables are connected by the relation $y=2-3x$.

   Read from your graph the value of x for which $y=0$.

   What value of x makes y assume the value -4 ? What value of y makes x assume the value $-0{\cdot}5$?

8. Plot the graph for the range $x=-4$ to $x=+4$ of the relation $y=2x-1$. Also on the same axes and to the same scale plot the graph of the relation $y=2x-2$.

   What do you notice about these graphs ? Would you expect the graph of the relation $y=2x$ to be a line parallel to the graph $y=2x-1$? Plot $y=2x$ and see.

9. Plot a graph to show the changes in the function $f(x)$ for the range $x=-2$ to $x=+4$, when $f(x)=3x-2$.

   From your graph find the value of x that makes $f(x)=4$, and the value of $f(x)$ when $x=-1$.

10. Draw axes and choose the same scale for x and y, suitable for a range from -4 to $+4$.

Indicate the points $P(2, 3)$, $Q(-2, 3)$, $R(-2, -3)$, $S(2, -3)$.

Mark the points $(0, 1)$ by A, $(-3, 0)$ by B, $(0, 0)$ by C, $(4, -2)$ by D.

Draw a circle centre C and radius CP. Verify that this circle passes through the points Q, R and S.

Write down the co-ordinates of the points where CA, CB and CD produced cut the circle.

Write down the co-ordinates of the points where AB, produced in both directions, cuts the circle.

11. Plot a graph for values of x from -6 to $+6$ of the function $f(x)=3x+4$. Also on the same axes plot a graph of $F(x)=4-3x$.

What do you observe about the way these lines slope ?

12. Plot graphs of the relations $f(x)=2x-3$ and $F(x)=1-2x$ for the range $x=-2$ to $x=+3$.

What do you observe about the way these lines slope ?

Compare the results of questions (11) and (12), and state how you would expect the graphs of the relations $f(x)=-5x+6$ and $F(x)=5x+3$ to slope.

13. Plot the graph of the relation $y=4x-3$ for the range $x=-5$ to $x=5$. Write down the co-ordinates of the points where $x=4$, 3, 0, -2 and -5 respectively. Write down also the co-ordinates of the points where $y=17$, 5, 1, -7 and -15 respectively.

Exercise 13f

1. On the same axes and to the same scale, plot the graphs of the relations $y=3$ and $y=-3$.

What do you notice about these graphs, and how would you expect the graphs of $x=2$ and $x=-2$ to lie ?

2. Plot the graphs of $y=3x+2$ and $y=3x-1$ on the same axes and with the same scales. Why are the graphs parallel ?

3. Which of the following pairs of relations will give rise to parallel graphs ?

(a) $y=2x-7$, $y=7x-2$.	(b) $y=5x+4$, $y=5x-3$.
(c) $y=6-x$, $y=1-x$,	(d) $y=3-2x$, $y=3+2x$.

4. Which of the following pairs of relations will give rise to parallel graphs ?

(a) $2y=6x+5$,
$3y=9x-2$.

(b) $5y=3x-1$,
$3y=5x+4$.

(c) $y=7-2x$,
$4y=9-8x$.

(d) $2y=5-3x$,
$4y=6x-1$.

5. Which of the following relations give rise to graphs passing through the origin ?

(1) $y=3x-2$. (2) $y=5x$. (3) $y=4-3x$. (4) $y=-2x$.
(5) $7y=x$. (6) $2y=-11x$. (7) $y+1=x$. (8) $3y-4x+2=0$.
(9) $y=9$. (10) $y=0$.

6. What is the equation of the graph parallel to $y=2x-5$ which passes through the origin ? Plot both graphs.

7. What is the equation of the graph parallel to $2y=4-3x$ which passes through the origin ? Plot both graphs.

8. Plot on the same axes and to the same scale the graphs of the relations : (a) $y=x-2$; (b) $y=2x-2$; (c) $y=-3x-2$, for the range $x=-3$ to $x=+3$.

At what point does each graph cut the y axis ?

9. Plot the graphs of the relations :

(a) $y=2x$; (b) $y=3x+1$; (c) $y=-1+4x$,

for the range $x=-5$ to $x=+5$.

Read from your graphs the intercepts on the y axis, and compare your results with the constant terms in the given relations.

What effect has the constant term on the point where the graph cuts the x axis ?

10. What intercept will the graph of $y=x+2$ make on the y axis ? Plot the graph and verify your answer.

11. What intercept will the graph of $3y=9-8x$ make on the y axis ? Plot the graph and verify your answer.

12. What are the intercepts made on the two axes by the graph of $5y=4x-7$? Plot the graph and verify your answer.

13. What are the intercepts made on the two axes by the graph of $4y+3x+6=0$? Plot the graph and verify your answer.

14. Write down the increase in x and the increase in y between the points (2, 4) and (3, 7).

If the increase in y is divided by the increase in x, what information is obtained about the line joining the two points ?

15. What are the gradients of the straight lines which pass through the following pairs of points?

(*a*) (1, 3) and (2, 11);
(*b*) (4, 5) and (7, 20);
(*c*) (4, 9) and (6, 12).

Which way do all these lines slope?

16. Find the gradients of the straight lines which pass through the following pairs of points:

(*a*) (2, 6) and (5, 3);
(*b*) (4, 26) and (7, 8);
(*c*) (1, 8) and (4, 3).

Which way do all these lines slope?

17. Use the same axes and scales to plot the graphs of:

(*a*) $y=3x-2$; (*b*) $y=\frac{3}{2}x+1$; (*c*) $y=3-x$,

for the range $x=-4$ to $x=+4$.

Obtain the gradient of each line from the graph, and compare your results with the coefficients of x in the given relations.

What effect has the constant term on the gradient?

18. Draw the graphs of the relations:

(*a*) $y=6x-5$; (*b*) $y=6x-2$; (*c*) $y=-6x-2$,

for the range $x=-3$ to $x=+3$.

What do you observe about the gradients of graphs (*a*) and (*b*)?

What explanation can you give for the result just observed?

Why would you not expect graph (*c*) to be parallel to (*a*)?

What statement can you make about the gradients of (*a*) and (*c*)?

19. What is the gradient of the graph of $2y=5x+1$?

Plot the graph and verify your answer.

20. What is the gradient of the graph of $9y=7-6x$?

Plot the graph and verify your answer.

21. What is the gradient of the graph of $3y-7x+5=0$?

Plot the graph and verify your answer.

22. Write down the intercept on the y axis and the gradient of the graph of $2x+3y+4=0$.

Check your answers by plotting the graph.

23. What are the values of a and b if the graph of $5y=ax+b$ has a gradient of 1·6 and passes through the point (1, 2)?

24. Find the equations of the three straight lines which pass through the three pairs of points given in question 15.
25. Find the equations of the three straight lines which pass through the three pairs of points given in question 16.

Exercise 13g

1. On the same axes and to the same scale, draw graphs of the relations $y=2x-1$ and $y=x+1$.

   From your graphs obtain the values of x and y at the point of intersection.

   Substitute these values in the two equations. What do you notice ?
2. Draw graphs of the relations $y=3x+5$ and $y=1-x$.

   Show that the values of x and y at the point of intersection of these two straight lines satisfy both equations.
3. On the same axes and to the same scale, plot the graphs of the relations $y=5x-3$ and $y=3x-5$, for the range $x=-2$ to $x=+4$.

   What are the values of x and y at the point where these graphs intersect, and what meaning can be attached to these values ?
4. By drawing the graphs of the relations $y=x-7$ and $y=2x-10$, solve these simultaneous equations.
5. Solve the following pair of simultaneous equations by drawing their graphs : $y=3x-1$; $y=-2x-9$.
6. Express the equations $5y-3x=25$ and $6x=-y-6$ in the form $y=ax+b$. Solve these equations graphically.
7. Express y in terms of x in the equations:

   $$5x+4y-4=0; \qquad x-12y-20=0.$$

   Draw the graphs and solve the equations.
8. By a graphical method, solve the following simultaneous equations :

   (a) $y=2x$; $3y=4x+1$.
   (b) $2y=5x+5$; $2y=5-4x$.
   (c) $7y=3x+3$; $4y=-x-1$.
   (d) $6y=2x-15$; $4y=6x-3$.
   (e) $3y=x$; $y=-x$.
   (f) $x-2y-3=0$; $7x-4y-16=0$.
   (g) $5x-3y-1=0$; $2x+y-7=0$.
9. Draw the graphs of $2y-5x=7$ and $15x-6y=-4$.

   Are there any values of x and y that satisfy both equations ?

10. Which of the following pairs of simultaneous equations can be solved ?

$$(a)\ y=4x-3,\quad y=3x+4. \qquad (b)\ y=2x-5,\quad y=2x+1.$$

Check your answer by drawing the graphs of each set.

11. Which of the following simultaneous equations are inconsistent ? Solve graphically the equations which are not inconsistent:

$$(a)\ y=-3x+7,\quad y=3x-2. \qquad (b)\ 2y=3x+7,\quad y=3x-2. \qquad (c)\ y=3x+7,\quad y=3x-2.$$

12. One of the following pairs of simultaneous equations is not independent and another is inconsistent. Solve the remaining pair graphically:

$$(a)\ 6y=2x+3,\quad 5x=10y-2. \qquad (b)\ 3y-15x+12=0,\quad 5x=4+y. \qquad (c)\ 6y=1-2x,\quad 9y+3x=-1.$$

13. Give values to a and b such that the simultaneous equations are (i) not independent, (ii) inconsistent.

$$3y=11x+5,$$
$$y=ax+b.$$

14. Find the equation of the straight line passing through the point (2, 5) and parallel to the line $y=-3x+1$.

15. Find the equation of the line passing through the point of intersection of $y=2x-11$ and $y+6x=1$, and also passing through the origin.

16. Plot the graphs of $2y=9x-2$ and $y+x+12=0$.

What is the point of intersection ? Find the equation of the straight line which passes through this point of intersection and the point $(-1, -3)$.

17. Draw the graph of $5x+11y+a=0$ if it is given that it passes through the point of intersection of the graphs of $3x+7y-2=0$ and $2x-y-7=0$.

CHAPTER XIV

Factors and Factorisation

87. In arithmetic we learn how to factorise numbers. We know from the multiplication tables that $3\times 7=21$. Thus 21 can be written as the product of the two numbers 3 and 7, which are called its **factors.**

Similarly, we know that $66=6\times 11$. But $6=2\times 3$, so that $66=2\times 3\times 11$. These three factors of 66 cannot be split up any further, and so are called **prime factors.**

An algebraic expression may or may not have factors. The expression x^2-y^2, for example, is the product of $x+y$ and $x-y$, so that these two expressions are its factors. The expression $2a+3b$, on the other hand, has no factors.

When an expression is written as the product of factors it is said to be **factorised**, or **resolved into factors.** Various devices may be used to factorise an expression.

If an expression is to be written as the product of two factors, then **if one of the factors is known, the other can be obtained by division.**

The simplest case of factorising in algebra is that of a single term.

Thus,
$$3xy=3\times x\times y,$$
$$12ab=2\times 2\times 3\times a\times b,$$
$$14mn^2=2\times 7\times m\times n\times n.$$

A factor may contain more than one letter.

Thus,
$$35x(2x-y)=5\times 7\times x\times (2x-y).$$

Highest Common Factor

88. In arithmetic the **Highest Common Factor (H.C.F.)** of two or more numbers is the largest number which will divide exactly into each. It is obtained as the product of all the factors which are common to all the given numbers.

Thus,
$$42=2\times 3\times 7,$$
$$90=2\times 3\times 3\times 5,$$
$$132=2\times 2\times 3\times 11.$$

The factors which are common to all the given numbers are 2 and 3. Thus the H.C.F. of 42, 90, 132 is 6. Similarly in

algebra, **the H.C.F. of two or more expressions is obtained as the product of all the factors which are common to all the given expressions.**

Thus,
$$2abc = 2\times a\times b\times c,$$
$$10b^2c = 2\times 5\times b\times b\times c,$$
$$14a^2b = 2\times 7\times a\times a\times b,$$

The factors which are common to all the above are 2 and b. Thus the H.C.F. of $2abc$, $10b^2c$, $14a^2b$ is $2b$.

Example 1. Find the H.C.F. of:
$$12xy(x+y),\ 28y^2(x+y),\ 88y(x+y)^2.$$

Factorising,
$$12xy(x+y) = 2\times 2\times 3\times x\times y\times (x+y),$$
$$28y^2(x+y) = 2\times 2\times 7\times y\times y\times (x+y),$$
$$88y(x+y)^2 = 2\times 2\times 2\times 11\times y\times (x+y)\times (x+y).$$

The common factors are 2, 2, y, $(x+y)$. and thus the H.C.F. is $4y(x+y)$.

Factorising by Removing the H.C.F.

89. So far we have considered the factorisation of single terms. We shall now learn how to factorise expressions containing more than one term.

In § 10 it was shown that $a(3x-2y-4z)=3ax-2ay-4az$. Since the expression $3ax-2ay-4az$ is equal to the product of the two quantities a and $3x-2y-4z$, it follows that these two quantities are the factors of the expression. Thus **when all the terms of an expression contain a common factor, the expression may be factorised by writing this term outside a bracket, and writing inside the bracket the result of dividing each term by the common factor.**

Example 2. Factorise $3bp+6bq$.

Each term of the expression is divisible by 3 and also by b. Hence the common factor of the two terms is $3b$.

Thus,
$$3bp = 3b\times p,$$
$$6bq = 3b\times 2q,$$
$$\therefore\ 3bp+6bq = 3b(p+2q).$$

It is now seen that **the first step in factorising expressions such as the above** is to find the product of all the factors which are common to all the terms of the expression, that is **to find the H.C.F. of the terms of the given expression.**

Example 3. Factorise $4x^2-12xy$.

The H.C.F. of the terms $4x^2$ and $12xy$ is $4x$.

Writing $4x$ outside the bracket, and dividing each of the terms of the expression by $4x$, we obtain :

$$4x^2-12xy=4x(x-3y).$$

Note.—$4x$ and $(x-3y)$ are factors of the expression $4x^2-12xy$, because when multiplied together the result is the given expression.

Example 4. Factorise $2ax+3ay-5a$.

The H.C.F. of the three terms is a. Writing this outside the bracket, we obtain :

$$2ax+3ay-5a=a(2x+3y-5).$$

Example 5. Factorise $6a^3b^2-15a^2b^4+9a^4b$.

We proceed as follows :

The highest number which is a factor of all the coefficients 6, 15, 9 is 3.

The highest power of a which is a factor of each term is a^2.

The highest power of b which is a factor of each term is b.

Hence $3a^2b$ is a factor of each term.

Thus, $\quad 6a^3b^2-15a^2b^4+9a^4b=3a^2b(2ab-5b^3+3a^2)$.

The student should verify that the product of $3a^2b$ and $(2ab-5b^3+3a^2)$ is $6a^3b^2-15a^2b^4+9a^4b$.

The questions in Exercise 14a, page 215, should now be attempted.

90. **Example 6.** Factorise $a(p+q)-3b(p+q)+2c(p+q)$.

We notice that each of the three terms contains the factor $(p+q)$.

To simplify the reasoning, we may represent the quantity $(p+q)$ by the single letter z.

The expression then becomes $az-3bz+2cz$.

Writing the common factor z outside the bracket,

$$\begin{aligned}\text{the expression}&=z(a-3b+2c)\\&=(p+q)(a-3b+2c),\end{aligned}$$

since z represents the quantity $(p+q)$.

Example 7. Factorise $6(a+2b)^2+8a(a+2b)-6b(a+2b)$.

We may denote the quantity $(a+2b)$ by the letter z.

The expression now becomes $6z^2+8az-6bz$, which has a factor $2z$.

$$\therefore \quad 6z^2+8az-6bz=2z(3z+4a-3b).$$

Substituting the value $a+2b$ for z,

$$\text{the expression}=2(a+2b)[3(a+2b)+4a-3b]$$
$$=2(a+2b)(3a+6b+4a-3b)$$
$$=2(a+2b)(7a+3b).$$

Verification by multiplying out both sides.

The pupil should verify that:

$$6(a+2b)^2+8a(a+2b)-6b(a+2b)$$
$$=6(a^2+4ab+4b^2)+8a^2+16ab-6ab-12b^2$$
$$=14a^2+34ab+12b^2;$$

and that $2(a+2b)(7a+3b)$ also equals $14a^2+34ab+12b^2$.

Thus the product of the factors found is equal to the original expression.

Verification by substitution.

A shorter form of check may frequently be applied in examples on factorisation. For any chosen values of the letters, the product of the factors must be equal to the given expression.

Thus, if we substitute $a=1$, $b=2$, say, in the expression

$6(a+2b)^2+8a(a+2b)-6b(a+2b)$, it becomes

$$6(5)^2+8\times1\times5-6\times2\times5=6\times25+40-60=130.$$

Substituting the same values for a and b in $2(a+2b)(7a+3b)$, the latter becomes $2\times5\times13=130$.

The questions in Exercise 14b, page 216, should now be attempted.

Factorising by Grouping

91. It is sometimes necessary to form a preliminary grouping of some of the terms of an expression before the method of Examples 6 and 7 can be employed.

Consider the expression $mn(3m-2n)^2+21m-14n$.

The two final terms have the common factor 7, and can be written $7(3m-2n)$.

Thus the whole expression becomes, $mn(3m-2n)^2+7(3m-2n)$.

Following the method of the last examples, the quantity $(3m-2n)$ is now denoted by a single letter, z, say. The pupil should, however, be able to carry out this device mentally, thus obtaining the factors, $(3m-2n)[mn(3m-2n)+7]$,

i.e. $$(3m-2n)(3m^2n-2mn^2+7).$$

92. Consider the expression $ax+bx+ay+by$ . . . (1)

It is seen that the four terms of the expression have no common factor. It will, however, be noticed that the first pair of terms have the common factor x, and the second pair have the common factor y.

Thus we may write, $ax+bx+ay+by=x(a+b)+y(a+b)$. (2)

The expression has now been reduced to two terms having the common factor $(a+b)$.

We thus obtain, $ax+bx+ay+by=(a+b)(x+y)$ . . (3)

Steps (1), (2) and (3) of the reasoning above may be illustrated by Figs. 28, 29 and 30 respectively, where ax is represented by the area of a rectangle of length a and height x, and so on.

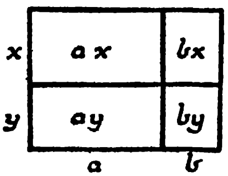

FIG. 28

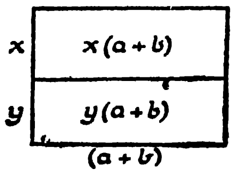

FIG. 29

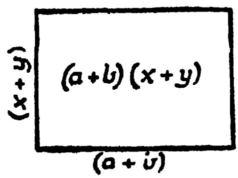

FIG. 30

It should be noticed that the same factors may be obtained by an alternative grouping of the terms of the expression.

Thus, $ax+bx+ay+by$ may be written

$$\begin{aligned} & ax+ay+bx+by \\ &=a(x+y)+b(x+y) \\ &=(x+y)(a+b). \end{aligned}$$

These factors are the same as those obtained in the first method, but in the reverse order. The steps of the reasoning in the second method of grouping are illustrated in Figs. 31, 32 and 33 respectively.

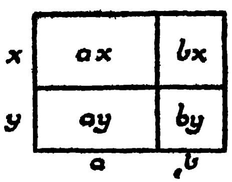

FIG. 31

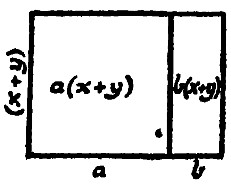

FIG. 32

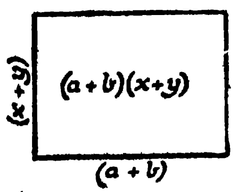

FIG. 33

Note.—We have factorised the expression $ax+bx+ay+by$ by grouping the terms in the order given, and also by grouping them from the new order $ax+ay+bx+by$. There is only one other way of grouping the terms into pairs, namely $(ax+by)+(bx+ay)$. It would not be possible, however, to obtain the factors of the expression from this grouping, since there is no factor common to the two terms in either bracket. Hence, of the three possible ways of grouping four terms into two pairs, two ways will result in the required factors, and one way will not.

Example 8. Factorise $a^2+2ab+5a+10b$.

$$\begin{aligned}\text{The expression} &= a(a+2b)+5(a+2b)\\ &= (a+2b)(a+5).\end{aligned}$$

Example 9. Factorise $12lp+6mq-8lq-9mp$.

The expression will not factorise if we group the terms in the order written. Rewriting,

$$\text{the expression} = 12lp-8lq+6mq-9mp.$$

The common factor of the first pair of terms is $4l$, and of the second pair is $3m$.

$$\begin{aligned}\text{Thus, the expression} &= 4l(3p-2q)+3m(2q-3p) \qquad . \quad . \quad \text{(A)}\\ &= 4l(3p-2q)-3m(3p-2q) \qquad . \quad . \quad \text{(B)}\\ &= (3p-2q)(4l-3m).\end{aligned}$$

Note.—The terms in line (A) do not appear to have any common factor. But $(2q-3p)=-(-2q+3p)=-(3p-2q)$,

$\therefore\ +3m(2q-3p)=-3m(3p-2q)$, thus giving the form in which the expression has been written in line (B).

The step $(x-y)=-(y-x)$ has frequent application in factorisation, and should be carefully noted.

With practice the pupil will be able to carry out the step in line (B) mentally, and proceed direct to the following line.

Example 10. Factorise $24ab^2+40bcd-16b^2c-60abd$.

The first step is to remove any factor which is common to all the terms.

Thus, the expression $=4b(6ab+10cd-4bc-15ad)$.

It is seen to be necessary to regroup the terms inside the bracket.

$$\begin{aligned}\text{Thus, the expression} &= 4b[6ab-4bc+10cd-15ad]\\ &= 4b[2b(3a-2c)+5d(2c-3a)]\\ &= 4b[2b(3a-2c)-5d(3a-2c)]\\ &\qquad\text{(See note to Ex. 9)}\\ &= 4b(3a-2c)(2b-5d).\end{aligned}$$

More Difficult Factorising by Grouping

93. An expression is not completely factorised until as many factors as possible have been obtained. When two factors of an expression have been found, it is sometimes possible to factorise further one or both of these factors.

Consider the expression $ab(x+3)+(b-a)(x+3)-x-3$.

This may be written $ab(x+3)+(b-a)(x+3)-(x+3)$. (A)

$=(x+3)(ab+b-a-1)$.

Factorising further the second bracket,

the expression $=(x+3)[b(a+1)-(a+1)]$ . . (B)

$=(x+3)(a+1)(b-1)$.

Note.—In line (A) $-(x+3)$ is the same as $-1\times(x+3)$, so that the last term in the second bracket of the following line is -1. Similarly, in line (B) the term $-(a+1)$ is the same as $-1\times(a+1)$, so that the last term in the final bracket of the following line is -1.

It may be necessary to remove the brackets in an expression before grouping is possible.

Example 11. Factorise $fg(a^2+b^2)+ab(g^2+f^2)$.

$$\begin{aligned}\text{The expression} &= a^2fg+b^2fg+abg^2+abf^2\\ &= a^2fg+abf^2+abg^2+b^2fg\\ &= af(ag+bf)+bg(ag+bf)\\ &= (ag+bf)(af+bg).\end{aligned}$$

94. When an expression contains more than four terms, they must be examined to discover the grouping which will yield factors.

Example 12. Factorise $8cx-3y(3a-b)+12ax-4bx-6cy$.

Removing the bracket,

the expression $=8cx-9ay+3by+12ax-4bx-6cy$.

Grouping the terms containing x and y separately,

$$\begin{aligned}\text{the expression} &= 8cx+12ax-4bx-9ay+3by-6cy\\ &= 4x(2c+3a-b)-3y(3a-b+2c)\\ &= (3a-b+2c)(4x-3y).\end{aligned}$$

The following rule is useful when there is difficulty in discovering the correct grouping for an expression.

If any letter occurs in the expression to the first power only, group all the terms containing that letter.

Example 13. Factorise $4axy+3abx^2-6a^2y^2+6bxy-9aby^2+2a^2x^2$.
It will be seen that the letter b occurs to the first power only.
Grouping the terms containing b,

$$\begin{aligned}\text{the expression}&=3abx^2+6bxy-9aby^2+4axy-6a^2y^2+2a^2x^2\\&=3b(ax^2+2xy-3ay^2)+2a(2xy-3ay^2+ax^2)\\&=(ax^2+2xy-3ay^2)(3b+2a).\end{aligned}$$

The questions in Exercise 14c, page 217, should now be attempted.

Factors of some Standard Expressions

95. The following identities are very important:

1. $\mathbf{a^2+2ab+b^2=(a+b)^2}$.
2. $\mathbf{a^2-2ab+b^2=(a-b)^2}$.
3. $\mathbf{a^2-b^2\quad=(a+b)(a-b)}$.
4. $\mathbf{a^3+b^3\quad=(a+b)(a^2-ab+b^2)}$.
5. $\mathbf{a^3-b^3\quad=(a-b)(a^2+ab+b^2)}$.

The pupil should multiply out the factors on the right-hand side of each identity, and verify that the expression on the left-hand side is obtained. These identities should be committed to memory.

Square of Sum and Square of Difference

96. The result $(a+b)^2=a^2+2ab+b^2$ can be put into words: **the square of the sum of two quantities is equal to the sum of their squares plus twice their product.**

Thus, $(2x+3y)^2=(2x)^2+2(2x)(3y)+(3y)^2=4x^2+12xy+9y^2$,
$(5a+4)^2=(5a)^2+2(5a)(4)+(4)^2=25a^2+40a+16$,
$(103)^2=(100)^2+2(100)(3)+(3)^2=10000+600+9=10609$.

97. The result $(a-b)^2=a^2-2ab+b^2$ can be put into words: **the square of the difference of two quantities is equal to the sum of their squares minus twice their product.**

Thus, $(7c-2d)^2=(7c)^2-2(7c)(2d)+(2d)^2=49c^2-28cd+4d^2$,
$(2x^2-5)^2=(2x^2)^2-2(2x^2)(5)+(5)^2=4x^4-20x^2+25$,
$(69)^2=(70-1)^2=(70)^2-2(70)(1)+(1)^2$
$=4900-140+1=4761$.

The pupil should aim at writing down directly the square of a sum and the square of a difference without the intermediate step.

The questions in Exercise 14d, page 218, should now be attempted.

98. *Examples on Identity* (1).

Example 14. Factorise $9x^2+30xy+25y^2$.

The expression may be written in the form of identity (1), namely, $(3x)^2+2(3x)(5y)+(5y)^2$.

If we write a for $3x$ and b for $5y$, this becomes

$$a^2+2ab+b^2$$
$$=(a+b)^2 \text{ from identity (1).}$$

Remembering that a stands for $3x$, and b for $5y$, this becomes

$$(3x+5y)^2.$$

Thus, $9x^2+30xy+25y^2=(3x+5y)^2$.

Example 15. Factorise $a^2+2ab+b^2+14(a+b)+49$.

This may be written $(a+b)^2+2(7)(a+b)+(7)^2$.

If we now write m for $(a+b)$, and n for 7, this becomes

$$m^2+2mn+n^2=(m+n)^2 \text{ from identity (1).}$$

Substituting $a+b$ for m, and 7 for n, this becomes $(a+b+7)^2$.

Thus, $a^2+2ab+b^2+14(a+b)+49=(a+b+7)^2$.

99. *Examples on Identity* (2).

Example 16. Factorise $100a^2-180ab+81b^2$.

The expression may be written $(10a)^2-2(10a)(9b)+(9b)^2$.

If we write x for $10a$, and y for $9b$, the expression becomes

$$x^2-2xy+y^2=(x-y)^2 \text{ from identity (2).}$$

Substituting $10a$ for x, and $9b$ for y, this becomes $(10a-9b)^2$.

Thus, $100a^2-180ab+81b^2=(10a-9b)^2$

Example 17. Factorise $121x^4-132x^2y^3+36y^6$.

The expression may be written $(11x^2)^2-2(11x^2)(6y^3)+(6y^3)^2$.

Following the method of the previous examples, the quantities $11x^2$ and $6y^3$ may now be denoted by single letters. The pupil should, however, be able to carry out this device mentally, thus obtaining $(11x^2-6y^3)^2$.

Example 18. Factorise $16p^2+24pq+9q^2-8p-6q+1$.

The first three terms may be written $(4p)^2+2(4p)(3q)+(3q)^2$

$$=(4p+3q)^2 \text{ from identity (1).}$$

Hence the original expression $=(4p+3q)^2-8p-6q+1$

$$=(4p+3q)^2-2\times1\times(4p+3q)+(1)^2$$
$$=(4p+3q-1)^2 \text{ from identity (2).}$$

The questions in Exercise 14e, page 218, should now be attempted.

Difference of Two Squares

100. The result $a^2-b^2=(a+b)(a-b)$ can be put into words: **the difference of the squares of two quantities is equal to the product of the sum and the difference of the two quantities.**

Examples on Identity (3).

Example 19. Factorise $49a^2-16b^2$.

The expression may be written $(7a)^2-(4b)^2$

$$=(7a+4b)(7a-4b) \text{ from identity (3).}$$

Thus, $49a^2-16b^2=(7a+4b)(7a-4b)$.

Example 20. Factorise $125x^8-80y^2$.

This is not at present in the form of identity (3). It is seen, however, that each coefficient is divisible by 5.

Thus, the expression $=5(25x^8-16y^2)$

$$=5[(5x^4)^2-(4y)^2]$$
$$=5(5x^4+4y)(5x^4-4y) \text{ from identity (3).}$$

Example 21. Factorise $(a^2+b^2)^2-(2ab)^2$.

Using identity (3), the expression $=(a^2+b^2+2ab)(a^2+b^2-2ab)$.

It will now be seen that these factors are the expressions contained in identities (1) and (2) respectively.

Hence the original expression $=(a+b)^2(a-b)^2$.

It is sometimes necessary to group the terms in an expression before an identity can be used.

Example 22. Factorise $9u^2+4v^2-16w^2-12uv+40w-25$.

The expression $=(9u^2-12uv+4v^2)-(16w^2-40w+25)$

$$=[(3u)^2-2(3u)(2v)+(2v)^2]-[(4w)^2-2(4w)5+(5)^2]$$
$$=(3u-2v)^2-(4w-5)^2 \text{ from identity (2)}$$
$$=[(3u-2v)+(4w-5)][(3u-2v)-(4w-5)] \text{ from identity (3)}$$
$$=(3u-2v+4w-5)(3u-2v-4w+5).$$

101. In the following examples the device used to put an expression into the form of identity (3) should be carefully noted.

Example 23. Factorise $x^4+x^2y^2+y^4$.

We know from identity (1) that $x^4+2x^2y^2+y^4=(x^2+y^2)^2$.

Thus we may rewrite the given expression as follows:

$$\begin{aligned} x^4+x^2y^2+y^4 &= x^4+2x^2y^2+y^4-x^2y^2 \\ &= (x^2+y^2)^2-(xy)^2 \\ &= (x^2+y^2+xy)(x^2+y^2-xy) \text{ from identity (3).} \end{aligned}$$

The result, $\mathbf{x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2)}$ is an important identity and should be committed to memory.

Example 24. Factorise $40x^2+4y^2-28xy-30x-25$.

The terms $4y^2-28xy$, that is $(2y)^2-2(2y)(7x)$, require a third term of $(7x)^2$, that is $49x^2$, to complete the form of the expression in identity (2). Hence the first term of the original expression, $40x^2$, must be written $49x^2-9x^2$.

Thus, the expression

$$\begin{aligned} &= 49x^2-28xy+4y^2-9x^2-30x-25 \\ &= (49x^2-28xy+4y^2)-(9x^2+30x+25) \\ &= [(7x)^2-2(7x)(2y)+(2y)^2]-[(3x)^2+2(3x)(5)+(5)^2] \\ &= (7x-2y)^2-(3x+5)^2 \\ &= [(7x-2y)+(3x+5)][(7x-2y)-(3x+5)] \\ &= (10x-2y+5)(4x-2y-5). \end{aligned}$$

Example 25. Factorise $25x^2-9y^2+30x-24y-7$.

The terms containing x, namely $25x^2+30x$, may be written

$$(5x)^2+2\times 3(5x).$$

To put this in the form of identity (1), we must add the term $(3)^2$, that is 9.

So that the expression should not be altered, we must also subtract 9.

Thus, the expression

$$\begin{aligned} &= 25x^2+30x+9-9y^2-24y-7-9 \\ &= 25x^2+30x+9-9y^2-24y-16 \\ &= 25x^2+30x+9-(9y^2+24y+16) \\ &= [(5x)^2+2(5x)(3)+(3)^2]-[(3y)^2+2(3y)(4)+(4)^2] \\ &= (5x+3)^2-(3y+4)^2 \\ &= [(5x+3)+(3y+4)][(5x+3)-(3y+4)] \\ &= (5x+3y+7)(5x-3y-1). \end{aligned}$$

Example 26. Factorise $25x^4+14x^2y^2+9y^4$.

The extreme terms can be written $(5x^2)^2$ and $(3y^2)^2$.

On reference to identity (1), we see that the middle term required to make an exact square is $2(5x^2)(3y^2)$, that is $30x^2y^2$.

Thus, the given expression may be written :

$$\begin{aligned}25x^4+14x^2y^2+9y^4&=25x^4+30x^2y^2+9y^4-16x^2y^2\\&=(5x^2+3y^2)^2-(4xy)^2\\&=(5x^2+3y^2+4xy)(5x^2+3y^2-4xy).\end{aligned}$$

The questions in Exercise 14f, page 219, should now be attempted.

Sum and Difference of Two Cubes

102. *Examples on Identity* 4.

Example 27. Factorise $8a^3+125$.

$$\begin{aligned}\text{The expression}&=(2a)^3+(5)^3\\&=(2a+5)[(2a)^2-(2a)(5)+(5)^2] \text{ from identity (4)}\\&=(2a+5)(4a^2-10a+25)\end{aligned}$$

$\therefore \quad 8a^3+125=(2a+5)(4a^2-10a+25)$.

Example 28. Factorise $135a^3x^{10}y^5+320a^{15}x^4y^2$.

This is not at present in the form of identity (4). It is seen, however, that each term is divisible by $5a^3x^4y^2$.

Thus, the expression

$$\begin{aligned}&=5a^3x^4y^2[27x^6y^3+64a^{12}]\\&=5a^3x^4y^2[(3x^2y)^3+(4a^4)^3]\\&=5a^3x^4y^2[(3x^2y+4a^4)\{(3x^2y)^2-(3x^2y)(4a^4)+(4a^4)^2\}]\\&\qquad\text{from identity (4)}\\&=5a^3x^4y^2(3x^2y+4a^4)(9x^4y^2-12a^4x^2y+16a^8).\end{aligned}$$

Example 29. Factorise $a^3+3a^2b+3ab^2+b^3$.

The extreme terms, if taken together, may be factorised by means of identity (4), while the middle terms have the common factor $3ab$.

$$\begin{aligned}\text{Thus, the expression}&=a^3+b^3+3a^2b+3ab^2\\&=(a+b)(a^2-ab+b^2)+3ab(a+b)\\&=(a+b)[(a^2-ab+b^2)+3ab]\\&=(a+b)(a^2+2ab+b^2)\\&=(a+b)(a+b)^2\\&=(a+b)^3.\end{aligned}$$

The identity which has been established, namely

$$\mathbf{(a+b)^3=a^3+3a^2b+3ab^2+b^3}$$

is important, and should be committed to memory.

Example 30. Factorise $a^3+b^3+c^3-3abc$.

We first rewrite the expression so as to make use of the identity established in the previous example.

$$a^3+b^3+c^3-3abc=a^3+3a^2b+3ab^2+b^3+c^3-3abc-3a^2b-3ab^2$$
$$=(a+b)^3+c^3-3abc-3a^2b-3ab^2.$$

The sum of the two cubes may be factorised by identity (4), while the remaining three terms have the common factor $3ab$.

Thus, the R.H.S. $=[(a+b)+c][(a+b)^2-c(a+b)+c^2]-3ab(c+a+b)$
$$=(a+b+c)(a^2+2ab+b^2-ca-bc+c^2)-3ab(a+b+c)$$
$$=(a+b+c)(a^2+2ab+b^2-ca-bc+c^2-3ab)$$
$$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

The identity which has been established, namely

$$\mathbf{a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}$$

should be committed to memory.

103. *Examples on Identity* (5).

Example 31. Factorise $343x^3-27y^3$.

The expression $=(7x)^3-(3y)^3$

$\therefore$ $=(7x-3y)[(7x)^2+(7x)(3y)+(3y)^2]$ from identity (5)

$=(7x-3y)(49x^2+21xy+9y^2)$.

Thus, $343x^3-27y^3=(7x-3y)(49x^2+21xy+9y^2)$.

Example 32. Factorise $7c^8d-448c^2d^7$.

Removing the common factor $7c^2d$,

the expression $=7c^2d[c^6-64d^6]$ . . . . . . (A)
$$=7c^2d[(c^3)^2-(8d^3)^2]$$
$$=7c^2d[(c^3+8d^3)(c^3-8d^3)]$$
$$=7c^2d[\{c^3+(2d)^3\}\{c^3-(2d)^3\}]$$
$$=7c^2d(c+2d)(c^2-2cd+4d^2)(c-2d)(c^2+2cd+4d^2).$$

The pupil should note that, after line (A), an alternative way of proceeding is to write

$c^6-64d^6=(c^2)^3-(4d^2)^3$, and then to use identity (5).

The pupil should show that this procedure will give the same factors as those found above, but that the working is more diffi-

cult. It will be found that where there is a choice of writing an expression as the difference of two squares or as the difference of two cubes, the easier method is to proceed by the difference of two squares.

Example 33. Factorise $a^3-3a^2b+3ab^2-b^3$.

The extreme terms, if taken together, may be factorised by means of identity (5), while the middle terms have the common factor $3ab$.

Thus, the expression
$$\begin{aligned}&=a^3-b^3-3a^2b+3ab^2\\&=(a-b)(a^2+ab+b^2)-3ab(a-b)\\&=(a-b)(a^2+ab+b^2-3ab)\\&=(a-b)(a^2-2ab+b^2)\\&=(a-b)(a-b)^2\\&=(a-b)^3.\end{aligned}$$

This identity, namely
$$\mathbf{(a-b)^3=a^3-3a^2b+3ab^2-b^3}$$
is important, and should be memorised.

Example 34. Factorise $xy(x-y)+x^2(x-2)+y^2(2-y)$.

The above expression does not appear to be related to any standard form. The first step is to remove the brackets, thus obtaining $x^2y-xy^2+x^3-2x^2+2y^2-y^3$.

On grouping so as to obtain the form of known identities,

the expression
$$\begin{aligned}&=(x^3-y^3)-2(x^2-y^2)+(x^2y-xy^2)\\&=(x-y)(x^2+xy+y^2)-2(x+y)(x-y)+xy(x-y)\\&=(x-y)[(x^2+xy+y^2)-2(x+y)+xy]\\&=(x-y)[(x^2+2xy+y^2)-2(x+y)]\\&=(x-y)[(x+y)^2-2(x+y)]\\&=(x-y)(x+y)(x+y-2).\end{aligned}$$

The questions in Exercise 14g, page 220, should now be attempted.

Exercise 14a

Find the H.C.F. of:

1. $4x^3, 12x^2, 8x$.
2. $2x^3y, 6x^2y^2, 10x^3y^3$.
3. $14a^3b, 35a^2b^2, 21b^3$.
4. $9x^4y^3, 12x^2y^4, 21x^2y^2$.
5. $a(a+b), b(a+b), c(a+b)$.
6. $x^2(x-y), xy(x-y), (x-y)$.

7. $x(a+b)$, $x^2(a+b)^2$, $x(a+b)^3$.
8. $ab(x-y)$, $a^2b(x-y)$, $b^2a(x-y)^2$.
9. $21x(a-b)$, $18xy(a-b)$, $33x^2(a-b)^2$.
10. $20ab(p-2q)$, $8ab^2(p-2q)$, $14a^2b(p-2q)$.

Resolve into factors :

11. x^2+bx.
12. $ax+xb$.
13. $ax+x$.
14. $ab-b^2$.
15. a^2b-a^3.
16. $2ax-2xb$.
17. $10d-20cd$.
18. $bcx+bdx$.
19. a^3x-2ax.
20. $14ax+21bx$.
21. $5ax^2-10x^3$.
22. ax^2c-ax^2d.
23. a^2b+ab.
24. $2ab^3-a^2b^2$.
25. $5acx+15cx$.
26. $18ab^2c+12a^2dc$.
27. $22a^3bc-33a^2c$.
28. $x^3y^2-xy^3$.
29. $3x^4y^4-7x^5y^2$.
30. $2a^2bd^2+3ab^3$.
31. $17p^3q^3-51p^2q^2$
32. $39x^2-52$.
33. $a^3-a^2b+ab^2$.
34. $-3ab-2b^3c+ab^2$.
35. $3a+3b+3c$.
36. $-9x^3y^2+15x^2y^3-21xy^3$.
37. $57x^6-19m^6x^2+38xy^6$.
38. $2ba^2+2ab^2+2b^3$.
39. $6x^3+18x^2+30x$.
40. $6ax^2+14ax-2a$.
41. $12a^7b^6-36a^6b^6-48a^3b^5c$.
42. $35x^3y-77x^2y-63xy$.
43. $a^2x^5+a^2x^3y^2+a^2xy^3$.
44. $m+23mx-7x^2m$.
45. $7x^3y^3z-3xy^3z^3-4x^3yz^3$.
46. $xyz-xzw-xyw$.
47. $-a^5-a^4+a^3$.

Use factors to find the value of :

48. $17\times59+17\times41$.
49. $23\times67-23\times55$.
50. $31\times96+14\times31$.
51. $47\times103-97\times47$.
52. $\frac{3}{7}$ of $109+\frac{3}{7}$ of 31.
53. $\frac{2}{5}$ of $67+\frac{3}{5}$ of 67.
54. $\frac{8}{11}$ of $162-\frac{8}{11}$ of 85.
55. 7% of £329+7% of £471.
56. 6% of £866−6% of £216.
57. 4% of £570+6% of £570.

Exercise 14b

Resolve into factors, and check your result by substitution :

1. $x(a+3)+y(a+3)$.
2. $x(a+b)+y(a+b)$.
3. $x(a+b)-y(a+b)$.
4. $(a-b)-y(a-b)$.
5. $p(a-b)-q(a-b)$.
6. $x(a+b)-3(a+b)$.
7. $x^2(m+n)+y^2(m+n)$.
8. $3x(x^2+y^2)+y^2(x^2+y^2)$.
9. $(p+q)t-6(p+q)$.
10. $y(h+k)-(h+k)x$.
11. $a(x-2y)+(x-2y)b$.
12. $-9(3-2x)+y(3-2x)$.

Resolve into factors, and check your results by multiplying out both sides :

13. $2xy(3x^2-2y^2)-3z(3x^2-2y^2)+z(3x^2-2y^2)$.
14. $p^2q(3pq-1)-(3pq-1)-x(3pq-1)$.
15. $ab(x^2+y)+ab(x+y^2)$.

16. $(ab-1)7m-7m(2a+3b)$.
17. $(x+y)^2+3(x+y)$.
18. $(a-b)-(a-b)^2$.
19. $(m-n)^2-2(m-n)$.
20. $(x+y)^2+a(x+y)$.
21. $(ab-c)^3+d(ab-c)^2$.
22. $(x+y)^2+2x(x+y)+4y(x+y)$.
23. $4(x-2y)^2+7y(x-2y)-3x(x-2y)$.
24. $3(2a+b)^2-2a(2a+b)+b(2a+b)$.
25. $3n(m-n)+2(m-n)^2-m(m-n)$.
26. $4a(3p-q)^2-6ap(3p-q)+10aq(3p-q)$.
27. $(c^3-d)^2+c^2(c^3-d)+d(c^3-d)$.

Exercise 14c

Resolve into factors :

1. $ac+ad+bc+bd$.
2. $x^2+ax+bx+ab$.
3. $ac-bc-ad+bd$.
4. $ab-ac-b^2+bc$.
5. $ac+ad-bc-bd$.
6. $x^2-yx-yz+zx$.
7. $xa+ay-xb-yb$.
8. $a^2-ac+ba-bc$.
9. $p^2+pq-p-q$.
10. $bxy+b^2-2xy-2b$.
11. $x^2y+3x^2+ay+3a$.
12. $a^2c-bc-5a^2+5b$.
13. $x^3y^3-cxy^2+dx^2y-cd$.
14. $a^4+ba^2+a^2c+bc$.
15. $a^4b-a^2b^2-ca^2+cb$.
16. $6ax+4x+9ya+6y$.
17. $12a^2+9xa-12xy-16ay$.
18. $p^3q^3-p^2qr-spq^2+sr$.
19. $a^4b^4+5ab^2+c^2db^2a^3+5c^2d$.
20. x^4+2x^3+2x+4.
21. x^3-x^2+x-1.
22. $2x^4-6x^3+ax-3a$.
23. $3m(x+3)+a(x+3)-x-3$.
24. $p(a+b)+qa+qb-2(a+b)$.
25. $x(a^2-b)-y(a^2-b)-a^2+b$.
26. $ax+bx-ay-cy+cx-by$.
27. $ac-bc+c^2+ay-by+cy$.
28. $a^3+ab^2-a^2b-b^3-a^2c-cb^2$.
29. $x(a+b)^2-ay-by$.
30. $3(x-y)^2-mx+my$.
31. $a(b-c)+a^2-bc$.
32. $a^2+bc+a(b+c)$.
33. $x^2+yz-x(z+y)$.
34. $ab(ab+c-d)-cd$.
35. $p^2+3q-p(q+3)$.
36. $x^2y^2-ab+xy(b-a)$.
37. $ab(p^2+q^2)+pq(a^2+b^2)$.
38. $(a^2+b^2)xy-ab(x^2+y^2)$.
39. $pq(m^2+n)-m(p^2+nq^2)$.
40. $a^2(b-1)-a(b-1)$.
41. $mn(p^2+1)+p(m^2+n^2)$.
42. $pq(x+1)+(p+q)(x+1)+x+1$.
43. $x^2(x-1)-x^2+x-(x-1)^2$.
44. $axy-bxy-(x+y)(a-b)+a-b$.
45. $x^3-x^2-x(x-1)+(x+1)(x-1)$.
46. $mn(p+q)-cp-cq-(m-cn)(p+q)$.
47. $ab(x^2+y^2)+(pa+qb)(x^2+y^2)+pq(x^2+y^2)$.
48. $m^2(m+1)+m(m+1)^2-(2m^2+1)(m+1)$.

Exercise 14d

Write down the squares of:

1. $x+y$.
2. $a+2$.
3. $3+d$.
4. x^2+1.
5. $2a+b$.
6. $c+3$.
7. $5x+3y$.
8. $7a+5b$.
9. $c-d$.
10. $x-y$.
11. $a-5$.
12. $4-m$.
13. a^2-1.
14. $3p-q$.
15. $m-4n$.
16. $6x-11y$.
17. $8+3a$.
18. $9g-1$.
19. $5h+4k$.
20. $1+xy$.
21. $2ab-3$.
22. $7xy+2z$.
23. x^2+1.
24. x^2+y^2.
25. x^2-y^2.
26. $3a^2-b^2$.
27. $4c^2-d$.
28. $-2a+3c$.
29. $-5m-6n$.
30. $8x^2+3y^2$.
31. $6-5a^2$.
32. x^2y-2y.
33. $11a^2b+3b^2$.
34. $9c^2+4d^3$.
35. $-3a^2-5ab$.
36. $7x^2y+8xy$.
37. $4a^2b-3ab^2$.
38. $12xy-5a^3$.
39. a^3+b^3.
40. a^3-b^3.
41. $8xyz-3$.
42. $7a^2b-4c^3$.
43. 104.
44. 109.
45. 98.
46. 39.
47. 203.
48. 307.
49. 61.
50. 199.
51. 78.

Exercise 14e

Resolve into factors:

1. $x^2+2ax+a^2$.
2. x^2+4x+4.
3. $x^2+10x+25$.
4. $4x^2+12x+9$.
5. $25x^2+10x+1$.
6. $a^2x^2+8ax+16$.
7. $a^2-2ay+y^2$.
8. y^2-6y+9.
9. $y^2-20y+100$.
10. $4y^2-28y+49$.
11. $81y^2-18y+1$.
12. $36y^2-60y+25$.
13. $p^2q^2-2pqr+r^2$.
14. $x^2+2abx+a^2b^2$.
15. $1-10c+25c^2$.
16. $121y^2+66y+9$.
17. $m^2+14am+49a^2$.
18. $a^2b^2-4ab+4$.
19. $a^2x^2+2ax+1$.
20. $36x^2+60ax+25a^2$.
21. $ab^2-2abc+ac^2$.
22. $c^2x^2+2abcx+a^2b^2$.
23. $p^4+2p^2q^2+q^4$.
24. $t^4-2t^2k^2+k^4$.
25. $4x^4-12x^2y^2+9y^4$.
26. $9y^4+30y^2z^2+25z^4$.
27. $a^4x^2+26a^2x+169$.
28. $m^4x^4+2m^2c^2x^2+c^4$.
29. $x^4y^4-2cx^2y^2+c^2$.
30. $144x^2+168a^2x+49a^4$.
31. $49h^4-14h^2y+y^2$.
32. $a^2b^4-2ab^2c+c^2$.
33. $x^4+4x^2y^2+4y^4$.
34. $1-2m^3n^2+m^6n^4$.
35. $(a+b)^2+2(a+b)+1$.
36. $3m^2-12lm+12l^2$.
37. $(a+1)^2-2(a+1)c+c^2$.
38. $(x+y)^2+2z(x+y)+z^2$.
39. $(m+n)^2x^2+2(m+n)x+1$.
40. $8a^2-24ab+18b^2$.
41. $4(p-q)^2+12s(p-q)+9s^2$.
42. $14x(a+b)c+49c^2x^2+(a+b)^2$.
43. $x^2+2xy+y^2-2(x+y)+1$.
44. $a^2+4a+4+2c(a+2)+c^2$.

45. $p^2+2pq+q^2+6(p+q)+9$. **46.** $x^2-2xy+y^2-2a(x-y)+a^2$.
47. $9(a+b)^2x^2+12(a+b)x+4$. **48.** $(m-n)^2-10(m-n)+25$.
49. $25+30k+9k^2+12(3k+5)m+36m^2$.
50. $(p+q)^2-6p+9-6q$.
51. $m(m+2n)+n^2$.
52. $x^2y^2+2xyz+z^2+16m(xy+z)+64m^2$.
53. $64a^4-112a^2b^2+49b^2$.
54. $4a^2-4a+1-4ab+2b+b^2$.
55. $x^4+2(2a+b)x^2+4a^2+4ab+b^2$.
56. $ax(ax+2p+2s)+p^2+2ps+s^2$.
57. $9m^2+n^2-6m-2n+6mn+1$.
58. $u^2v^2+(a+b)(2uv+a+b)$.

Exercise 14f

Resolve into factors :

1. x^2-y^2. **2.** a^2-1. **3.** m^2-4.
4. $49-y^2$. **5.** $(ab)^2-36$. **6.** $25x^2-y^2$.
7. $1-a^2b^2$. **8.** $16m^2-81n^2$. **9.** $(a^2)^2-y^2$.
10. $169x^2y^2-4$. **11.** $50y^2-2$. **12.** b^3-b.
13. $9a^2b^2-c^2d^2$. **14.** $12-3m^2$. **15.** m^4-4.
16. $p^2q^4-t^4$. **17.** $121a^4-1$. **18.** $x^4y^6-25z^2$.
19. c^6-16d^2. **20.** $100-k^8$. **21.** $9a^8-49b^4$.
22. $5a^2-20b^2$. **23.** ab^2-ad^2. **24.** $256x^2y^4-81z^2$.
25. $2-72m^4n^6$. **26.** x^3-xy^2. **27.** a^4-a^2.
28. $(a+b)^2-1$. **29.** $4(x-y)^2-9(xy)^2$. **30.** $(a^2+b)^2-25c^2$.
31. $(x+y)^2-(a-b)^2$. **32.** $3c^2-432(l+m)^2$. **33.** ax^5-axb^2.
34. x^4-y^4. **35.** a^4-1. **36.** a^3b-ab^3.
37. $(a^2+2ab)^2-b^4$. **38.** $(4p^2-4pq)^2-q^4$.
39. $a^2+b^2-c^2+4c-4+2ab$.
40. $9m^2-6mn+n^2-1$. **41.** $4x^2y^2-9+4xyz+z^2$.
42. $(c^2+4d^2)^2-(4cd)^2$. **43.** $(p^2+5pq+q^2)^2-9p^2q^2$.
44. $16(x-y)^3-(x-y)$. **45.** $(a-x)y^2-25(a-x)$.
46. $(x^4+2x^2y^2+y^4)-x^2y^2$. **47.** $a^4+a^2y^2+y^4$.
48. $k^4+4k^2t^2+16t^4$. **49.** $x^4-27x^2y^2+81y^4$.
50. $9x^4-10x^2y^2+y^4$. **51.** p^4+4y^4.
52. a^4+64b^4. **53.** $4c^4+9d^4-16c^2d^2$.
54. $x^2+2x+1-y^2-4y-4$. **55.** $c^2+6c+9-d^2-2d-1$.
56. p^2+2p-q^2-6q-8. **57.** $a^2-4y^2-10a-4y+24$.
58. $9m^2-n^2-8n+12m-12$. **59.** $25a^2-45-9x^2-20a+42x$.
60. $16x^2-40y+24x-25y^2-7$. **61.** $3l^2+4lm+m^2-4l-4$.
62. $9a^2+16b^2-30ab+6b-1$. **63.** $24x^2+28xy-60x+4y^2-9$.

Use factors to find the value of:

64. $(134)^2-(133)^2$.
65. $(372)^2-(367)^2$.
66. $(421)^2-(179)^2$.
67. $(239)^2-121$.
68. $(57\cdot6)^2-(57\cdot2)^2$.
69. $(99\cdot9)^2-0\cdot01$.
70. $(71\cdot3)^2-1\cdot69$.

Exercise 14g

Resolve into factors:

1. x^3+y^3.
2. x^3+2^3.
3. a^3+1.
4. $m^3+(2n)^3$.
5. $(ab)^3+y^3$.
6. 3^3+t^3.
7. $64+27x^3$.
8. m^3-n^3.
9. p^3-1.
10. $(2a)^3-1$.
11. $2(x)^3-2$.
12. 2^3-m^3.
13. a^3-125.
14. $8x^3-27$.
15. $250-16m^3$.
16. $8(ax)^3-1$.
17. $3m^3+24$.
18. xy^3-x.
19. a^4+a.
20. $(x^2)^3-(y^2)^3$.
21. $(a-1)^3+8$.
22. $(a+b)^3-(a-b)^3$.
23. $(x+y)^3+(x-y)^3$.
24. $(2a-b)^3-(2a)^3$.
25. x^3+x-2.
26. a^3+a-10.
27. $8a^3+b^3+2a+b$.
28. c^6-d^6.
29. $x^2a^6-729x^8$.
30. $a^3+a^2-ab+b^2+b^3$.
31. $64+a^6$.
32. $(8c^3)^2-1$.
33. $8x^3+12x^2y+6xy^2+y^3$.
34. $128a^6+2$.
35. $a^3-6a^2b+12ab^2-8b^3$.
36. $a^3x^3+9a^2x^2b+27axb^2+27b^3$.
37. $1-3b+3b^2-b^3$.
38. $x^3y^3-3x^2y^2+3xy-1$.
39. $a^3+8b^3-6ab+1$.
40. $8x^3-y^3+27z^3+18xyz$.

CHAPTER XV

Factors of Quadratic Expressions

When the Coefficient of x^2 is Unity

104. A **quadratic expression** is one which contains a letter to the second power but to no higher power. Thus, $4x^2-3x+11$ is a quadratic expression in x.

We shall now consider methods for factorising quadratic expressions by studying the product of factors which give quadratic expressions.

Consider the product $(x+3)(x+4)$.

$$\begin{array}{l} x+3 \\ \underline{x+4} \\ x^2+3x \\ \underline{\quad\;\; 4x+12} \\ \underline{\underline{x^2+7x+12}} \end{array}$$

Thus, $x^2+7x+12=(x+3)(x+4)$.

Examining the factors on the R.H.S., we observe:

1. Each factor commences with the term x, which is the square root of the first term of $x^2+7x+12$.
2. The remaining terms in the factors, that is 3 and 4, are such that (*a*) their product is 12, which is the constant term in $x^2+7x+12$, and (*b*) their sum is 7, which is the coefficient of the middle term in $x^2+7x+12$.

We now use this method to factorise $x^2+7x+10$.

1. Each factor must commence with x.
2. The remaining terms in the factors must be such that (*a*) their product is 10, and (*b*) their sum is 7.

The only two numbers satisfying these conditions are 2 and 5.

Thus the factors are $(x+2)$ and $(x+5)$.

$$\therefore \quad x^2+7x+10=(x+2)(x+5).$$

In the above example all the terms are positive.

Consider next the product $(x+7)(x-3)$.

$$\begin{array}{r} x+7 \\ x-3 \\ \hline x^2+7x \\ -3x-21 \\ \hline x^2+4x-21 \end{array}$$

Thus, $x^2+4x-21=(x+7)(x-3)$.

Examining the factors on the R.H.S., we observe:

1. Each factor commences with x.
2. The remaining terms in the factors, that is 7 and -3, are such that (*a*) their product is -21, which is the constant term in $x^2+4x-21$, and (*b*) their *algebraic* sum is 4, which is the coefficient of the middle term in $x^2+4x-21$.

Example 1. Factorise $x^2+3x-40$.

1. Each factor must commence with x.
2. The remaining terms in the factors must be such that (*a*) their product is -40, hence one must be positive and one negative, and (*b*) their *algebraic sum* (which is numerically equal to their *difference*) is 3.

The only two numbers satisfying these conditions are 8 and -5.

Thus, $x^2+3x-40=(x+8)(x-5)$.

Example 2. Factorise $x^2-3x-10$.

1. Each factor must commence with x.
2. The remaining terms in the factors must be such that (*a*) their product is -10, hence one must be positive and one negative, and (*b*) their *algebraic sum* (which is numerically equal to their *difference*) is -3.

The only two numbers satisfying these conditions are 2 and -5.

Thus, $x^2-3x-10=(x+2)(x-5)$.

105. **Example 3.** Factorise $-x^2+15x-44$.

When factorising an expression such as the above, which contains the term $-x^2$, it is convenient to rewrite the whole expression first, with the minus sign outside a bracket.

Thus, $-x^2+15x-44=-(x^2-15x+44)$.

Factorising $x^2-15x+44$.

1. Each factor must commence with x.
2. The remaining terms in the factors must be such that (*a*) their product is 44, and (*b*) their algebraic sum is -15.

The only two numbers satisfying these conditions are -4 and -11.

Thus the factors are $(x-4)$ and $(x-11)$.

$$\therefore \quad -x^2+15x-44=-(x-4)(x-11).$$

The factors may be left as above. Alternatively, the minus sign may be included in **one** of the factors (but not both), thus :

$$(4-x)(x-11) \text{ or } (x-4)(11-x).$$

The questions in Exercise 15a, page 232, should now be attempted.

106. Example 4. Factorise y^4-7y^2-18.

This can be written $(y^2)^2-7(y^2)-18$, and is thus a quadratic in y^2, the y^2 taking the place of the x in the previous examples.

1. Each factor must commence with y^2, which is the square root of the first term in y^4-7y^2-18.
2. The remaining terms in the factors must be such that (*a*) their product is -18, hence one must be positive and one negative, and (*b*) their algebraic sum, which is numerically equal to their difference, is -7.

The only two numbers satisfying these conditions are 2 and -9.

Thus, $$y^4-7y^2-18=(y^2+2)(y^2-9)$$
$$=(y^2+2)(y+3)(y-3).$$

Example 5. Factorise $x^4-11ax^2+30a^2$.

1. Each factor must commence with x^2.
2. The remaining terms in the factors must be such that (*a*) their product is $30a^2$, and (*b*) their algebraic sum is $-11a$.

Thus the terms are $-5a$ and $-6a$.

Thus, $x^4-11ax^2+30a^2=(x^2-5a)(x^2-6a)$.

The questions in Exercise 15b, page 233, should now be attempted.

When the Coefficient of x^2 is not Unity

The Cross Method

107. In the previous examples the coefficient of the first term in the quadratic expression has been unity.

Consider the product $(2x+3)(7x+5)$.

$$\begin{array}{r} 2x+\ 3 \\ 7x+\ 5 \\ \hline 14x^2+21x \\ +10x+15 \\ \hline 14x^2+31x+15 \\ \hline \end{array}$$

Thus, $14x^2+31x+15=(2x+3)(7x+5)$.

Examining the factors on the R.H.S., we observe:

1. The coefficients of the first terms in the factors, that is 2 and 7, are such that their product is 14, which is the coefficient of the first term in $14x^2+31x+15$.
2. The remaining terms in the factors, that is 3 and 5, are such that their product is 15, which is the constant term in $14x^2+31x+15$.
3. These coefficients are so arranged that if we multiply the first in each bracket by the second in the other bracket, the algebraic sum of these two products (that is $2\times5+7\times3$, as shown by dotted lines) is equal to 31, which is the coefficient of the middle term in $14x^2+31x+15$.

The conditions which these coefficients must satisfy can be readily expressed by the following device:

$$\begin{array}{c|c} 2 & 3 \\ \hline 7 & 5 \end{array}$$

The coefficients of the first terms in the factors, that is 2 and 7, are written in the L.H. quadrants of the cross, and must be such that their product is equal to the coefficient of the first term of the quadratic expression. The remaining terms of the factors, that is 3 and 5, are written in the R.H. quadrants, and must be such that their product is equal to the constant term in the quadratic expression. The coefficients must be so chosen that the algebraic sum of the diagonal products, that is $2\times5+7\times3$, is equal to the coefficient of the middle term of the quadratic expression.

Example 6. Factorise $4x^2+7x+3$.

Proceeding as indicated in the last example, the coefficients in the L.H. quadrants must have a product equal to 4. These can be 2 and 2, or 4 and 1. The coefficients in the R.H. quadrants must have a product 3. These can be 3 and 1, or 1 and 3.

Thus there are four possible crosses :

$$\begin{array}{c|c} 2 & 3 \\ \hline 2 & 1 \end{array}, \qquad \begin{array}{c|c} 2 & 1 \\ \hline 2 & 3 \end{array}, \qquad \begin{array}{c|c} 4 & 3 \\ \hline 1 & 1 \end{array}, \qquad \begin{array}{c|c} 4 & 1 \\ \hline 1 & 3 \end{array}.$$

These possibilities can be shown on one cross :

$$\begin{array}{cc|cc} 2 & 4 & 3 & 1 \\ \hline 2 & 1 & 1 & 3 \end{array}$$

The additional condition to be satisfied is that the algebraic sum of the diagonal products is 7. This condition is satisfied only by

$$\begin{array}{c|c} 4 & 3 \\ \hline 1 & 1 \end{array}.$$

Thus, $4x^2+7x+3=(4x+3)(x+1)$.

Note.—To include all possibilities, we should have started with the cross

$$\begin{array}{ccc|cc} 2 & 4 & 1 & 3 & 1 \\ \hline 2 & 1 & 4 & 1 & 3 \end{array}.$$

We observe that all the conditions are satisfied by the additional group

$$\begin{array}{c|c} 1 & 1 \\ \hline 4 & 3 \end{array}$$

$$\therefore \quad 4x^2+7x+3=(x+1)(4x+3).$$

It is seen, however, that these factors are those already obtained, but in reverse order. **It is not necessary, therefore, to invert the pairs of factors in both L.H. and R.H. quadrants. All the possibilities are included if the pairs of factors are inverted on one side only.**

Example 7. Factorise $10x^2-x-21$.

We fill in the cross in such a way that the L.H. quadrants give a product 10, and the R.H. quadrants give a product 21.

The trials are
$$\begin{array}{cc|cccc} 10 & 5 & 21 & 1 & 7 & 3 \\ \hline 1 & 2 & 1 & 21 & 3 & 7 \end{array}.$$

As the constant term is negative, the numbers in the R.H. quadrants must have opposite signs. Hence one diagonal product

will be positive and the other negative, so that their *algebraic sum* is numerically equal to their *difference*.

The condition that the difference of the diagonal products is 1 is satisfied by the group $\dfrac{5}{2}\Big|\dfrac{7}{3}$.

Since the middle coefficient is -1, the diagonal products must be -15 and $+14$. Thus the group with the correct signs affixed is

$$\frac{5}{2}\Big|\frac{7}{-3}.$$

$$\therefore \quad 10x^2-x-21=(5x+7)(2x-3).$$

Example 8. Factorise $18x^2-59x+35$.

We fill in the cross in such a way that the L.H. quadrants give a product 18, and the R.H. quadrants give a product 35.

The trials are $\dfrac{18 \quad 9 \quad 6}{1 \quad 2 \quad 3}\Big|\dfrac{35 \quad 1 \quad 7 \quad 5}{1 \quad 35 \quad 5 \quad 7}$

As the constant term is positive, the numbers in the R.H. quadrants must have like signs. Hence the numerical value of the algebraic sum of the diagonal products is also the arithmetical sum.

The condition that the sum of the diagonal products is 59 is satisfied by the group $\dfrac{9}{2}\Big|\dfrac{7}{5}$.

Since the middle coefficient is -59, the diagonal products must be -45 and -14. Thus the group with the correct signs affixed is

$$\frac{9}{2}\Big|\frac{-7}{-5}.$$

$$\therefore \quad 18x^2-59x+35=(9x-7)(2x-5).$$

108. It is seen from the two previous examples that **the trial figures may be placed in the quadrants of the cross without signs, provided that it be remembered that a positive constant term implies the *sum* of the diagonal products, while a negative constant term implies the *difference* of the diagonal products.**

When the correct group has been chosen, the signs can be readily affixed according to the sign of the coefficient of the middle term. (This statement assumes the coefficient of x^2 to be positive.)

Example 9. Factorise $14-6x^2-17x$.

Arranging the terms in descending order, with the minus sign outside the bracket (see § 105),

$$14-6x^2-17x=-(6x^2+17x-14).$$

Factorising $6x^2+17x-14$, the trials are

$$\begin{array}{cc|cccc} 6 & 3 & 14 & 1 & 7 & 2 \\ \hline 1 & 2 & 1 & 14 & 2 & 7 \end{array}$$

As the constant term is negative, the middle coefficient, that is $+17$, is obtained by subtracting the diagonal products.

The condition that the difference of the diagonal products is 17 is satisfied by the group $\begin{array}{c|c} 3 & 2 \\ \hline 2 & 7 \end{array}$.

Since the middle coefficient is $+17$, the group with signs affixed is

$$\begin{array}{c|c} 3 & -2 \\ \hline 2 & 7 \end{array}.$$

Thus the factors are $(3x-2)$ and $(2x+7)$.

$$\therefore\quad 14-6x^2-17x=-(3x-2)(2x+7)$$
$$=(2-3x)(2x+7).$$

The questions in Exercise 15c, page 233, should now be attempted.

Reducing the Number of Trials

109. The labour involved in working through the trials can usually be very much reduced if the principles used in the following examples are carefully noted.

Example 10. Factorise $16x^2+61x-12$.

The trials are $\begin{array}{ccc|cccccc} 16 & 8 & 4 & 12 & 1 & 6 & 2 & 4 & 3 \\ \hline 1 & 2 & 4 & 1 & 12 & 2 & 6 & 3 & 4 \end{array}$.

The middle coefficient, 61, is an odd number. Since this number is obtained by subtracting the diagonal products, one of these products must be odd and the other even.

Hence the trials $\begin{array}{cc|c} 8 & 4 & \\ \hline 2 & 4 & \end{array}$

can be rejected, since they would produce diagonal products which are both even. The L.H. quadrant is thus $\begin{array}{c|c} 16 & \\ \hline 1 & \end{array}$.

For the same reason, the even numbers in the top R.H. quadrant can be rejected. Thus the only trials left are $\begin{array}{c|cc} 16 & 1 & 3 \\ \hline 1 & 12 & 4 \end{array}$ and eighteen possible trials have been reduced to two.

The difference of the diagonal products is 61, so that the correct group is

$$\begin{array}{c|c} 16 & 3 \\ \hline 1 & 4 \end{array}.$$

As the middle coefficient is $+61$, the group with signs affixed is

$$\begin{array}{c|c} 16 & -3 \\ \hline 1 & 4 \end{array}.$$

$$\therefore \quad 16x^2+61x-12=(16x-3)(x+4).$$

Example 11. Factorise $28x^2-64x+33$.

The trials are $\begin{array}{ccc|cccc} 28 & 14 & 7 & 33 & 1 & 11 & 3 \\ \hline 1 & 2 & 4 & 1 & 33 & 3 & 11 \end{array}$.

The middle coefficient is even, hence the diagonal products are either both even or both odd. Obviously they cannot be both odd, hence they must be both even. Thus the numbers in the L.H. quadrants must be 14 and 2.

Hence the trials have been reduced to $\begin{array}{c|cccc} 14 & 33 & 1 & 11 & 3 \\ \hline 2 & 1 & 33 & 3 & 11 \end{array}$.

Since the sum of the diagonal products is 64, the correct group is

$$\begin{array}{c|c} 14 & 11 \\ \hline 2 & 3 \end{array}.$$

As the middle coefficient is -64, the group with the signs affixed is

$$\begin{array}{c|c} 14 & -11 \\ \hline 2 & -3 \end{array}.$$

$$\therefore \quad 28x^2-64x+33=(14x-11)(2x-3).$$

Example 12. Factorise $12x^2+7x-45$.

The trials are $\dfrac{\begin{array}{ccc|cccccc}12 & 6 & 3 & 45 & 1 & 15 & 3 & 9 & 5\end{array}}{\begin{array}{ccc|cccccc}1 & 2 & 4 & 1 & 45 & 3 & 15 & 5 & 9\end{array}}$.

As the middle coefficient is odd, the numbers 6, 2 in the L.H. quadrants are rejected.

It will next be noticed that the first and third terms are divisible by the prime number 3, so that at least one of the diagonal products must contain the factor 3. But the difference of the diagonal products is 7, which is *not* divisible by 3. Hence the other diagonal product must not be divisible by 3. Thus the factor 3 of the first term and the factor 9 of the last term must occur in the same diagonal.

The trials are reduced to $\dfrac{\begin{array}{cc|cc}12 & 3 & 1 & 5\end{array}}{\begin{array}{cc|cc}1 & 4 & 45 & 9\end{array}}$.

As the difference of the diagonal products is $+7$, the correct grouping, with signs, is $\dfrac{\begin{array}{c|c}3 & -5\end{array}}{\begin{array}{c|c}4 & 9\end{array}}$.

$$\therefore\quad 12x^2+7x-45=(3x-5)(4x+9).$$

Example 13. Factorise $6x^2-35x+50$.

The trials are $\dfrac{\begin{array}{cc|cccccc}6 & 2 & 50 & 1 & 25 & 2 & 10 & 5\end{array}}{\begin{array}{cc|cccccc}1 & 3 & 1 & 50 & 2 & 25 & 5 & 10\end{array}}$.

As the middle coefficient is odd, and the top L.H. quadrant is even, the top R.H. quadrant must be odd.

The trials are reduced to $\dfrac{\begin{array}{cc|ccc}6 & 2 & 1 & 25 & 5\end{array}}{\begin{array}{cc|ccc}1 & 3 & 50 & 2 & 10\end{array}}$.

It will next be noticed that the constant term is divisible by the prime number 5, so that at least one of the diagonal products must contain the factor 5. But the sum of the diagonal products is 35, which is divisible by 5. Hence the other diagonal product must also contain the factor 5. Thus both R.H. quadrants contain the factor 5.

The trials become $\dfrac{\begin{array}{cc|c}6 & 2 & 5\end{array}}{\begin{array}{cc|c}1 & 3 & 10\end{array}}$.

The correct grouping and sign is $\dfrac{\begin{array}{c|c}2 & -5\end{array}}{\begin{array}{c|c}3 & -10\end{array}}$.

$$\therefore\quad 6x^2-35x+50=(2x-5)(3x-10).$$

110. The pupil will find the following summary helpful:

1. **If the middle coefficient is an odd number, then one pair of quadrants taken diagonally must contain only odd numbers.** (See Example 10.)
2. **If the middle coefficient is even, then the diagonal products are both even or both odd.** (See Example 11.)
3. **If the coefficients of the first and last terms contain a common prime factor, and the middle term does not contain it, this factor must be in the same diagonal.** (See Example 12.)
4. **If the coefficients of the middle and last terms have a common prime factor, this factor must occur in both the R.H. quadrants.** (See Example 13.)
5. **If the coefficients of the first and middle terms have a common prime factor, this factor must occur in both the L.H. quadrants.**

The questions in Exercise 15d, page 234, should now be attempted.

Literal Coefficients

111. **Example 14.** Factorise $15y^4+(5b^2-3ab)y^2-ab^3$.

This is a quadratic in y^2.

The trials are $\dfrac{15\quad 5}{1\quad 3}\left|\dfrac{ab^3\quad 1\quad ab^2\quad b\quad ab\quad b^2\quad a\quad b^3}{1\quad ab^3\quad b\quad ab^2\quad b^2\quad ab\quad b^3\quad a}\right.$.

As the middle coefficient is to be $5b^2-3ab$, the correct grouping and sign is $\dfrac{5}{3}\left|\dfrac{-ab}{b^2}\right.$.

Thus, the expression$=(5y^2-ab)(3y^2+b^2)$.

112. **Example 15.** Factorise $39x^2-83ax+38a^2$.

As the middle coefficient contains the factor a, the sum of the diagonal products must be divisible by a. But since the third term is divisible by a, one at least of the diagonal products must contain the factor a. Hence both of the diagonal products must be divisible by a. Thus the factor a must occur in each of the R.H. quadrants. If this is borne in mind, we can consider the

coefficients to be 39, -83 and 38 while forming the trials, and insert the a after finding the correct grouping.

The trials are $\dfrac{39\quad 13}{1\quad 3}\Bigg|\dfrac{38\quad 1\quad 19\quad 2}{1\quad 38\quad 2\quad 19}$.

The correct grouping and sign is seen to be $\dfrac{13}{3}\Bigg|\dfrac{-19}{-2}$.

$$\therefore\quad 39x^2-83ax+38a^2=(13x-19a)(3x-2a).$$

We conclude that **any quadratic expression which contains a letter as a factor of the middle coefficient, and the square of that letter as a factor of either the first or third terms, can be more readily factorised by omitting that letter in the trials.**

113. If a quadratic expression with literal coefficients is to be factorised, the choice available in filling the quadrants of the cross is usually much more limited than when the quadratic has numerical coefficients.

Example 16. Factorise $y^2-b^2+5-6y+4b$.

The first step is to write the expression in the normal order. It can be written as a quadratic either in y or in b.

Writing the expression as a quadratic in y,

$$y^2-6y-b^2+4b+5.$$

The constant term, that is $-b^2+4b+5=-(b^2-4b-5)$, has only one pair of factors, namely, $-(b-5)(b+1)$.

Thus the trials for the quadratic in y are

$$\frac{1}{1}\Bigg|\frac{(b-5)\qquad\quad (b-5)(b+1)}{(b+1)\qquad\qquad 1}.$$

The correct grouping and sign is $\dfrac{1}{1}\Bigg|\dfrac{b-5}{-(b+1)}$.

$$\therefore\quad y^2-b^2+5-6y+4b=(y+b-5)(y-b-1).$$

Example 17. Factorise

$$6(3x-2y)^2+(a-5)(3x-2y)-(2a^2-13a+21).$$

Considering this as a quadratic in $(3x-2y)$, the coefficients are

$$6,\quad (a-5),\quad -(2a^2-13a+21).$$

The factors of $2a^2-13a+21$ are $(2a-7)(a-3)$.

Thus the trials are

$$\frac{6}{1}\ \frac{3}{2}\Bigg|\ \frac{(2a-7),}{(a-3)}\quad \frac{(a-3),}{(2a-7)}\quad \frac{(2a-7)(a-3),}{1}\quad \frac{1}{(2a-7)(a-3)}.$$

The correct grouping and sign is $\frac{3}{2}\Bigg|\frac{(2a-7)}{-(a-3)}$.

Thus, the expression $=[3(3x-2y)+(2a-7)][2(3x-2y)-(a-3)]$
$=(9x-6y+2a-7)(6x-4y-a+3)$.

114. The following result is instructive. Consider a quadratic expression in which the coefficients of the first term and the constant term are both odd. The trials must be $\frac{\text{odd}}{\text{odd}}\Bigg|\frac{\text{odd}}{\text{odd}}$.

Hence each diagonal product is odd, and thus both their sum and their difference must be even. Thus the middle coefficient *must* be even. Hence we see that a quadratic expression in which all three coefficients are odd *cannot* have any integral linear factors.

The questions in Exercise 15e, page 235, should now be attempted.

Exercise 15a

Factorise :

1. x^2+2x+1.	**2.** x^2+4x+3.	**3.** x^2+3x+2.
4. x^2+6x+5.	**5.** x^2-8x+7.	**6.** x^2-5x+6.
7. $x^2-14x+13$.	**8.** $x^2-9x+18$.	**9.** x^2-x-2.
10. x^2-2x-3.	**11.** x^2-x-6.	**12.** $x^2-3x-10$.
13. x^2+x-2.	**14.** $x^2+2x-15$.	**15.** $x^2+12x-28$.
16. x^2+x-30.	**17.** x^2+4x+4.	**18.** x^2+7x+6.
19. $x^2-7x+12$.	**20.** x^2-x-12.	**21.** $x^2-15x+56$.
22. $x^2-9x-22$.	**23.** $x^2+8x-33$.	**24.** $x^2-35x-36$.
25. x^2+x-72.	**26.** $x^2-14x+45$.	**27.** $x^2-12x-45$.
28. $x^2+10x+9$.	**29.** $x^2+10x-39$.	**30.** $x^2-24x+143$.
31. $y^2+8y+15$.	**32.** $-a^2+19a-88$.	**33.** $c^2+13c-68$.
34. $d^2-4d-77$.	**35.** $-x^2+9x+70$.	**36.** $b^2-19b-92$.
37. $m^2+22x+21$.	**38.** $x^2+17x+66$.	**39.** $a^2-17a+52$.
40. $-y^2+19y+42$.	**41.** $b^2+6b-27$.	**42.** $p^2-23p+102$.
43. $f^2+17f+72$.	**44.** $m^2-9m-112$.	**45.** $-h^2-h+20$.
46. $q^2+2q-63$.	**47.** $c^2-2c-35$.	**48.** $a^2+13a+30$.

49. $-x^2+18x+144$.
50. $-y^2+24y-63$.
51. $d^2+3d-40$.
52. $p^2+15p-54$.
53. $m^2+16m+28$.
54. $g^2-8g-84$.
55. $h^2-14h+48$.
56. $d^2+6d-112$.
57. $-b^2-20b-51$.
58. $c^2-21c+90$.
59. $x^2+x-156$.
60. $y^2-y-132$.
61. $p^2-22p+72$.
62. $300+13g-g^2$.
63. $f^2+17f+42$.
64. $p^2q^2+10pq+16$.
65. $x^2y^2-25xy+24$.
66. $(ab)^2-7ab-78$.
67. $m^2n^2+2mn-143$.
68. $h^2+18h-243$.
69. $y-y^2+380$.
70. $l^2m^2+50lm+49$.
71. $x^2-26x+153$.
72. $203-a^2-22a$.
73. $c^2+25c+126$.
74. $(ax)^2-20(ax)+75$.
75. $g^2+22g-279$.
76. $d^2+22d+57$.
77. $27b-182-b^2$.
78. $t^2-80t-81$.
79. $m^2+6m-391$.
80. $s^2+20s+91$.
81. $25q-q^2-126$.
82. $c^2d^2+9cd+20$.

Exercise 15b

Factorise :

1. x^4+11x^2+18.
2. y^4-4y^2-21.
3. $6a^2-8-a^4$.
4. c^4+12c^2-45.
5. $(a^2b^2)^2-5a^2b^2-36$.
6. $\left(\frac{y}{x}\right)^2+4\left(\frac{y}{x}\right)+4$.
7. $x^2+11ax+24a^2$.
8. $-y^4+3ay^2+28a^2$.
9. $p^4-12p^2q^2+27q^4$.
10. $(x+1)^2+5(x+1)+6$.
11. $(x-2)^2-9(x-2)+14$.
12. $(2x-1)^2-2(2x-1)-15$.
13. $-(x^2-3)^2+4(x^2-3)+12$.
14. $(x^2-2x)^2-2(x^2-2x)-3$.
15. $8-(a^3)^2-7a^3$.

Exercise 15c

Factorise :

1. $2x^2+3x+1$.
2. $2x^2+5x+2$.
3. $3x^2+5x+2$.
4. $3x^2+7x+2$.
5. $5x^2+21x+4$.
6. $4x^2+12x+5$.
7. $2x^2-5x-3$.
8. $3x^2-x-2$.
9. $5x^2-4x-1$.
10. $7x^2-34x-5$.
11. $3x^2-11x-4$.
12. $4x^2-4x-3$.
13. $2x^2-7x+3$.
14. $3x^2-8x+5$.
15. $7x^2-15x+2$.
16. $2x^2-5x+3$.
17. $5x^2-12x+7$.
18. $4x^2-12x+5$.
19. $2x^2+5x-3$.
20. $3x^2+2x-5$.
21. $4x^2+4x-3$.
22. $7x^2+11x-6$.
23. $5x^2+8x-4$.
24. $9x^2+6x-8$.
25. $6x^2+31x+5$.
26. $5x^2-6x-8$.
27. $3x^2-11x+6$.
28. $6x^2+13x-5$.
29. $7x^2-24x+9$.
30. $6x^2-11x-7$.
31. $7x^2+15x+2$.
32. $12x^2+17x+6$.
33. $5x^2-19x+12$
34. $-9x^2+13x+10$.
35. $3x^2+14x+11$.
36. $10x^2-21x+9$
37. $-8x^2-13x+6$.
38. $11x^2-4x-7$.
39. $8x^2-2x-15$.

40. $15x^2+11x-12$.
41. $5y^2+13y+6$.
42. $11a^2+24a+4$.
43. $13b^2-7b-6$.
44. $10m^2-9m-9$.
45. $8p^2-10p+3$.
46. $-15t^2+26t-8$.
47. $18c^2+7c-8$.
48. $11y^2+15y-14$.
49. $-6n^2-11n-4$.
50. $9s^2-12s+4$.
51. $18d^2-13d-11$.
52. $-15a^2+11a+14$.
53. $2q^2-15q+18$.
54. $4f^2-17f+15$.
55. $6h^2+25h+4$.
56. $5x^2-19x+12$.
57. $-16x^2-14x+15$.
58. $6a^2-2a-8$.
59. $16m-4m^2-15$.
60. $9g^2+21g+10$.
61. $18+9p-14p^2$.
62. $21t^2-16t+3$.
63. $6a^2+5a-21$.
64. $12-14y^2-13y$.
65. $13x^2+18x-16$.
66. $4b^2+17b+4$.
67. $6n^2+19n+8$.
68. $k+6-12k^2$.
69. $15w^2-32w+16$.
70. $21s^2+13s-18$.
71. $-c-10c^2+24$.
72. $12v^2-17v+6$.
73. $20d^2-13d-15$.
74. $9x^2+21x+10$.
75. $8w-16w^2+15$.
76. $22y^2+13y-12$.
77. $10a^2+13a+4$.
78. $16x^2-30x+9$.
79. $24-11p-28p^2$.
80. $2t^2+4t+2$.

Exercise 15d

Keeping in mind the summary given in § 110, factorise :

1. $8x^2+17x+2$.
2. $10x^2+21x+8$.
3. $4x^2-21x+20$.
4. $16x^2-37x+21$.
5. $24x^2+5x-14$.
6. $4x^2+3x-52$.
7. $28x^2-45x-7$.
8. $40x^2-17x-12$.
9. $12x^2+32x+21$.
10. $15x^2+22x+8$.
11. $7x^2-24x+20$.
12. $49x^2-112x+60$.
13. $24x^2+38x-25$.
14. $21x^2+4x-76$.
15. $-33x^2+10x+8$.
16. $13x^2-2x-48$.
17. $12x^2+19x+4$.
18. $-11x^2-50x-24$.
19. $36x^2-61x+20$.
20. $35x^2-68x-132$.
21. $44x^2+9x-5$.
22. $-25x^2+58x-16$.
23. $-16x^2-22x+15$.
24. $15x^2-33x-48$.
25. $44y^2+52y+15$.
26. $12t^2-37t-144$.
27. $-28m^2+68m-39$.
28. $39a^2+4a-20$.
29. $23h^2-87h+64$.
30. $32f^2+197f+30$.
31. $16x^2-49x-60$.
32. $-75c^2+20c+84$.
33. $15x^2+46x+35$.
34. $18x^2-91x+45$.
35. $-40x^2-49x+24$.
36. $14+41x-28x^2$.
37. $25x^2-90x-63$.
38. $77x^2-18x-72$.
39. $9-27x-22x^2$.
40. $18x^2+33x-121$.
41. $28x^2-60x+27$.
42. $8x^2+30x+25$.
43. $23x-7x^2-16$.
44. $9+48x^2-43x$.
45. $16t^2+40-133t$.
46. $-76x+32+45x^2$.
47. $20b^2-27-12b$.
48. $16p^2q^2+14pq-147$.
49. $-68x^2y^2+48xy+81$.
50. $28ab-44+57a^2b^2$.
51. $-11mn-28+24m^2n^2$.
52. $-5y^2-21y^4+4$.
53. $35c^2-12-8c^4$.
54. $120y^2-113y-16$.
55. $20\left(\frac{x}{y}\right)^2+48\left(\frac{x}{y}\right)+27$.
56. $15a^4+29a^2+8$.

57. $19x^2+8x^4-15.$

58. $34\dfrac{p}{q}+8+33\dfrac{p^2}{q^2}.$

59. $27\left(\dfrac{m}{n}\right)^4-60\left(\dfrac{m}{n}\right)^2+28.$

60. $36p^2q^2-36pq-55.$

61. $41a^2+14-28a^4.$

62. $14(xy)^4-65x^2y^2-25.$

63. $48\dfrac{a^2}{b^2}-28+55\left(\dfrac{a^2}{b^2}\right)^2$

64. $32-10h^2+22h.$

65. $175-28x^2+225x.$

Exercise 15e

Factorise :

1. $3x^2+4ax+a^2.$

2. $4x^2+(4+b^2)x+b^2.$

3. $5x^2-(5q+p)x+pq.$

4. $6x^2+(3n-2m)x-mn.$

5. $10x^2+(5b^2+2a^2)x+a^2b^2.$

6. $9y^2-(5a+9b)y+5ab.$

7. $14x^2+(7a^2-6)x-3a^2.$

8. $42x^2-(49a-6b^2)x-7ab^2.$

9. $-6(xy)^2-(3b-4)xy+2b$

10. $15x^4+26x^2y^2+8y^4.$

11. $2ax^2+(3a-2b)x-3b.$

12. $-3p^2x^2+(3p^2q+2)x-2q.$

13. $26y^4-(39c+10b)y^2+15bc$

14. $5ab^2+(4a-5c)b-4c$

15. $-11y^4+(11-abc)y^2+abc$

16. $10m^2n^2+(2x-5y)mn-xy.$

17. $7y^2x^2-(y^3+14)x+2y.$

18. $-13x^2y^4-(s+39t)xy^2-3st.$

19. $x^2-2x-(a^2+6a+8).$

20. $x^2-x-a^2+3a-2.$

21. $81t-10t^2-45.$

22. $-12p^2+25p+75.$

23. $y^2-8y-4b^2+12b+7.$

24. $12x-36+35x^2.$

25. $x^2+cx-b^2+3bc-2c^2.$

26. $-48+21t^2-8t.$

27. $3(x+y)^2+(3b+a)(x+y)+ab.$

28. $y^2+7by-a^2+3ab+10b^2.$

29. $-5(x+y)^2-(6a+3b)(x+y)-a^2+ab+2b^2.$

30. $3y^2-3b^2-y-5b-2.$

31. $6(x+a)^2+(b-5c)(x+a)-2b^2+c^2-bc$

32. $7(x-2a)^2-2(2b-c)(x-2a)-3b^2+8bc-5c^2.$

CHAPTER XVI

Harder Fractions

115. In order to reduce a fraction to its lowest terms, it is necessary to divide numerator and denominator by every factor which is common to both, that is, by their H.C.F.

Example 1. Reduce $\dfrac{12x^3yz^2}{15x^2y^3z}$ to its lowest terms.

The H.C.F. of the numerator and denominator is $3x^2yz$.

Thus, $$\frac{12x^3yz^2}{15x^2y^3z}=\frac{3x^2yz\times 4xz}{3x^2yz\times 5y^2}=\frac{4xz}{5y^2}.$$

Example 2. Reduce $\dfrac{13ab^2c}{39a^2b^3c}$ to its lowest terms.

The H.C.F. of the numerator and denominator is $13ab^2c$.

Thus, $$\frac{13ab^2c}{39a^2b^3c}=\frac{13ab^2c\times 1}{13ab^2c\times 3ab}=\frac{1}{3ab}.$$

Note.—The process of dividing numerator and denominator by a common factor is called cancelling that factor. When a factor is cancelled it gives rise to a factor unity in both numerator and denominator.

To make cancelling easier, the numerator and denominator should, if necessary, be factorised.

Example 3. Simplify $\dfrac{15x^2yz}{25x^2z-10xyz}$.

Factorising the denominator,

$$\text{the fraction}=\frac{15x^2yz}{5xz(5x-2y)}$$
$$=\frac{5xz\times 3xy}{5xz(5x-2y)} \quad . \quad . \quad . \quad . \quad \text{(A)}$$
$$=\frac{3xy}{5x-2y}.$$

After a little practice line (A) may be omitted.

Example 4. Simplify $\dfrac{14ab^3-21a^2bc}{35a^2b^2-7abc^2}$.

Factorising numerator and denominator,

$$\text{the fraction}=\frac{7ab(2b^2-3ac)}{7ab(5ab-c^2)}.$$

$$=\frac{2b^2-3ac}{5ab-c^2}.$$

Example 5. Simplify

$$\frac{(a+2b)(3a-1)-(b-1)(a+2b)}{(b-3a)(ab-2)}.$$

Factorising the numerator,

$$\text{the fraction}=\frac{(a+2b)[(3a-1)-(b-1)]}{(b-3a)(ab-2)} \quad . \quad . \quad . \quad (A)$$

$$=\frac{(a+2b)(3a-1-b+1)}{(b-3a)(ab-2)}$$

$$=\frac{(a+2b)(3a-b)}{(b-3a)(ab-2)} \quad . \quad . \quad . \quad . \quad (B)$$

$$=\frac{(a+2b)(3a-b)}{-(3a-b)(ab-2)}$$

$$=\frac{(a+2b)}{-(ab-2)}$$

$$=\frac{a+2b}{2-ab}.$$

With a little practice line (A) may be omitted.

In line (B) it will be seen that the factor $(b-3a)$ in the denominator can be written as $-(3a-b)$, thus making it possible to cancel.

In the same way the student should notice that

$$\frac{1}{7-x}, \qquad \frac{1}{-(x-7)}, \qquad \frac{-1}{x-7} \quad \text{and} \quad -\frac{1}{x-7}$$

are all equal.

Similarly, $$\frac{2a-3b}{3y-x}=-\frac{2a-3b}{x-3y},$$

and $$\frac{a-b}{c-d}=\frac{b-a}{d-c}.$$

Example 6. Reduce $\dfrac{a^3b-ab^3}{a^3-b^3}$ to its lowest terms.

Factorising numerator and denominator,

the fraction
$$=\frac{ab(a^2-b^2)}{(a-b)(a^2+ab+b^2)}$$
$$=\frac{ab(a+b)(a-b)}{(a-b)(a^2+ab+b^2)}$$
$$=\frac{ab(a+b)}{a^2+ab+b^2}.$$

Example 7. Simplify $\dfrac{3x^2+7x-6}{6x^2+11x-10}$.

The fraction
$$=\frac{(3x-2)(x+3)}{(3x-2)(2x+5)}$$
$$=\frac{x+3}{2x+5}.$$

Example 8. Simplify
$$\frac{16x^4-81y^4}{(2x-3y)(20x^4+33x^2y^2-27y^4)}.$$

The fraction
$$=\frac{(4x^2+9y^2)(4x^2-9y^2)}{(2x-3y)(4x^2+9y^2)(5x^2-3y^2)}$$
$$=\frac{(4x^2+9y^2)(2x+3y)(2x-3y)}{(2x-3y)(4x^2+9y^2)(5x^2-3y^2)}$$
$$=\frac{2x+3y}{5x^2-3y^2}.$$

The questions in Exercise 16a, page 245, should now be attempted.

Multiplying and Dividing Fractions

116. Example 9. Simplify
$$\frac{x^2+ax}{xy-2y^2}\times\frac{xz-2yz}{xy+ay}\div\frac{xz}{y^3}.$$

Remembering that to divide by a fraction we can multiply by the fraction inverted,

the expression

$$=\frac{x^2+ax}{xy-2y^2}\times\frac{xz-2yz}{xy+ay}\times\frac{y^3}{xz}$$

$$=\frac{x(x+a)}{y(x-2y)}\times\frac{z(x-2y)}{y(x+a)}\times\frac{y^3}{xz}$$

$$=y.$$

Example 10. Simplify

$$\frac{27a^4-8a}{6a^2+5a-1}\times\frac{6a-1}{9a^2+6a+4}\div\frac{3a^2-2a}{a+1}.$$

The expression

$$=\frac{27a^4-8a}{6a^2+5a-1}\times\frac{6a-1}{9a^2+6a+4}\times\frac{a+1}{3a^2-2a}$$

$$=\frac{a(27a^3-8)}{(6a-1)(a+1)}\times\frac{6a-1}{9a^2+6a+4}\times\frac{a+1}{a(3a-2)}$$

$$=\frac{a(3a-2)(9a^2+6a+4)}{(6a-1)(a+1)}\times\frac{6a-1}{9a^2+6a+4}\times\frac{a+1}{a(3a-2)} \quad . \quad . \quad (A)$$

$$=1.$$

Notice carefully that **as all the factors** that occur in line (A) **cancel, the fraction becomes $\frac{1}{1}$, that is 1, and not zero.**

The questions in Exercise 16b, page 246, should now be attempted.

Adding and Subtracting Fractions

117. In Part I, Chapter VIII, we worked through some simple examples on the addition and subtraction of fractions. We now take some more difficult examples.

Method 1. When the denominators cannot be factorised.

Example 11. Simplify $\frac{4}{x-1}-\frac{5}{x-3}+\frac{3}{x+2}.$

The L.C.M. of the denominators is $(x-1)(x-3)(x+2)$.

The expression

$$=\frac{4(x-3)(x+2)-5(x-1)(x+2)+3(x-1)(x-3)}{(x-1)(x-3)(x+2)}$$

$$=\frac{4(x^2-x-6)-5(x^2+x-2)+3(x^2-4x+3)}{(x-1)(x-3)(x+2)}$$

$$=\frac{4x^2-4x-24-5x^2-5x+10+3x^2-12x+9}{(x-1)(x-3)(x+2)}$$

$$\frac{2x^2-21x-5}{(x-1)(x-3)(x+2)}.$$

Method 2. By writing each fraction as a mixed fraction. (See § 40.)

Example 12. Simplify $\frac{6a-5}{2a-3}-\frac{2a+3}{a+2}-\frac{3a-1}{3a+2}$.

The expression

$$=\frac{3(2a-3)+4}{2a-3}-\frac{2(a+2)-1}{a+2}-\frac{(3a+2)-3}{3a+2}$$

$$=3+\frac{4}{2a-3}-2+\frac{1}{a+2}-1+\frac{3}{3a+2}$$

$$=\frac{4}{2a-3}+\frac{1}{a+2}+\frac{3}{3a+2}$$

$$=\frac{4(a+2)(3a+2)+(2a-3)(3a+2)+3(2a-3)(a+2)}{(2a-3)(a+2)(3a+2)}$$

$$=\frac{12a^2+32a+16+6a^2-5a-6+6a^2+3a-18}{(2a-3)(a+2)(3a+2)}$$

$$=\frac{24a^2+30a-8}{(2a-3)(a+2)(3a+2)}.$$

Method 3. When the denominators can be factorised.

Example 13. Simplify $\frac{5}{6x^2-13x-5}+\frac{4}{12x^2-5x-3}$.

The expression $=\dfrac{5}{(3x+1)(2x-5)}+\dfrac{4}{(3x+1)(4x-3)}.$

The L.C.M. of the denominators is $(3x+1)(2x-5)(4x-3)$.

Denoting this by D,

the expression $=\dfrac{5(4x-3)+4(2x-5)}{D}$

$$=\frac{20x-15+8x-20}{D}$$

$$=\frac{28x-35}{(3x+1)(2x-5)(4x-3)}.$$

Example 14. Simplify

$$\frac{3}{6a^2-5ab-6b^2}-\frac{4}{14a^2-19ab-3b^2}+\frac{1}{21a^2+17ab+2b^2}.$$

The expression

$$=\frac{3}{(2a-3b)(3a+2b)}-\frac{4}{(2a-3b)(7a+b)}+\frac{1}{(3a+2b)(7a+b)}.$$

The L.C.M. of the denominators is $(2a-3b)(3a+2b)(7a+b)$.

Denoting this by D,

the expression $=\dfrac{3(7a+b)-4(3a+2b)+(2a-3b)}{D}$

$$=\frac{21a+3b-12a-8b+2a-3b}{D}$$

$$=\frac{11a-8b}{(2a-3b)(3a+2b)(7a+b)}.$$

Example 15. Simplify

$$\frac{2x-3y}{x^2-xy}+\frac{2x-y}{x^2-3xy+2y^2}-\frac{2y(2x-y)}{x^3-3x^2y+2xy^2}.$$

The expression $=\dfrac{2x-3y}{x(x-y)}+\dfrac{2x-y}{(x-y)(x-2y)}-\dfrac{4xy-2y^2}{x(x-y)(x-2y)}.$

The L.C.M. of the denominators is $x(x-y)(x-2y)$.

Denoting this by D,

the expression$=\dfrac{(2x-3y)(x-2y)+x(2x-y)-(4xy-2y^2)}{D}$

$$=\frac{2x^2-7xy+6y^2+2x^2-xy-4xy+2y^2}{D}$$

$$=\frac{4x^2-12xy+8y^2}{D}=\frac{4(x^2-3xy+2y^2)}{D}$$

$$=\frac{4(x-y)(x-2y)}{x(x-y)(x-2y)}=\frac{4}{x}.$$

Example 16. Simplify

$$\frac{1}{2(1-x)}+\frac{1}{2-x}+\frac{6x-7}{4(x^2-3x+2)}-\frac{1}{(4-2x)^2}.$$

In finding the L.C.M. of these denominators, the following points are important:

$$(1-x)=-(x-1),$$
$$(2-x)=-(x-2),$$
$$x^2-3x+2=(x-1)(x-2),$$
$$(4-2x)^2=[2(2-x)]^2=4(2-x)^2=4(x-2)^2.$$

The expression may therefore be written:

$$-\frac{1}{2(x-1)}-\frac{1}{x-2}+\frac{6x-7}{4(x-1)(x-2)}-\frac{1}{4(x-2)^2}.$$

The L.C.M. of the denominators is $4(x-1)(x-2)^2$.

Denoting this by D,

the expression

$$=\frac{-2(x-2)^2-4(x-1)(x-2)+(6x-7)(x-2)-(x-1)}{D}$$

$$=\frac{-2(x^2-4x+4)-4(x^2-3x+2)+6x^2-19x+14-x+1}{D}$$

$$=\frac{-2x^2+8x-8-4x^2+12x-8+6x^2-19x+14-x+1}{D}$$

$$=\frac{-1}{4(x-1)(x-2)^2}.$$

Example 17. Simplify $\dfrac{\dfrac{4d^2+9}{4d^2-9}-\dfrac{4d^2-9}{4d^2+9}}{\dfrac{2d+3}{2d-3}-\dfrac{2d-3}{2d+3}}.$

The numerator $=\dfrac{(4d^2+9)^2-(4d^2-9)^2}{(4d^2-9)(4d^2+9)}$ . . . . (A)

$$=\frac{16d^4+72d^2+81-(16d^4-72d^2+81)}{(4d^2-9)(4d^2+9)}$$

$$=\frac{144d^2}{(4d^2-9)(4d^2+9)}.$$

The denominator $=\dfrac{(2d+3)^2-(2d-3)^2}{(2d-3)(2d+3)}$ . . . . (B)

$$=\frac{4d^2+12d+9-(4d^2-12d+9)}{(2d-3)(2d+3)}$$

$$=\frac{24d}{(2d-3)(2d+3)}.$$

Hence the fraction$=\dfrac{144d^2}{(4d^2-9)(4d^2+9)}\div\dfrac{24d}{(2d-3)(2d+3)}$

$$=\frac{144d^2}{(4d^2-9)(4d^2+9)}\times\frac{(2d-3)(2d+3)}{24d}$$

$$=\frac{144d^2}{(2d+3)(2d-3)(4d^2+9)}\times\frac{(2d-3)(2d+3)}{24d}$$

$$=\frac{6d}{4d^2+9}.$$

Note.—The numerators in lines (A) and (B) could be factorised as the difference of two squares.

Substitution

118. We have previously used the method of substitution to check the accuracy of algebraic working. We now take an example of the same check in the simplification of fractions.

Example 18. Simplify

$$\left(\frac{2b}{a+b}-\frac{b}{a}\right)\div\left(1+\frac{a}{b-2a}\right)-\left(1-\frac{2b}{a+b}\right)\div\left(\frac{a}{b}+\frac{a}{a-2b}\right),$$

and check the result by the substitution $a=1$, $b=3$.

The expression

$$=\left(\frac{2ab-ab-b^2}{a(a+b)}\right)\div\left(\frac{b-2a+a}{b-2a}\right)-\left(\frac{a+b-2b}{a+b}\right)\div\left(\frac{a^2-2ab+ab}{b(a-2b)}\right)$$

$$=\left\{\frac{ab-b^2}{a(a+b)}\right\}\div\left(\frac{b-a}{b-2a}\right)-\left(\frac{a-b}{a+b}\right)\div\left\{\frac{a^2-ab}{b(a-2b)}\right\}$$

$$=\frac{b(a-b)}{a(a+b)}\times\frac{b-2a}{b-a}-\frac{a-b}{a+b}\times\frac{b(a-2b)}{a(a-b)} \quad . \quad . \quad . \quad . \quad . \quad \text{(A)}$$

$$=-\frac{b(b-2a)}{a(a+b)}-\frac{b(a-2b)}{a(a+b)}$$

$$=-\frac{b^2+2ab-ab+2b^2}{a(a+b)}=\frac{ab+b^2}{a(a+b)}$$

$$=\frac{b(a+b)}{a(a+b)}=\frac{b}{a}.$$

Substituting $a=1$, $b=3$, the original expression becomes

$$\left(\frac{2\times3}{1+3}-\frac{3}{1}\right)\div\left(1+\frac{1}{3-2\times1}\right)-\left(1-\frac{2\times3}{1+3}\right)\div\left(\frac{1}{3}+\frac{1}{1-2\times3}\right)$$

$$=\left(\frac{6}{4}-3\right)\div\left(1+1\right)-\left(1-\frac{6}{4}\right)\div\left(\frac{1}{3}-\frac{1}{5}\right)$$

$$=\left(-\frac{3}{2}\right)\div\left(2\right)-\left(-\frac{1}{2}\right)\div\left(\frac{2}{15}\right)=-\frac{3}{4}+\frac{15}{4}=3.$$

Substituting $a=1$, $b=3$ in the answer we obtain $\frac{3}{1}=3$.

Note.—The denominator $(b-a)$ which occurs in line (A) can be rewritten as $-(a-b)$, thus allowing the factor $(a-b)$ to be cancelled.

We may be required to substitute algebraic quantities.

Example 19. Substitute $x=a+1$ and $y=a-1$ in the expression

$$\left(\frac{1-y}{1+y}-\frac{1-x}{1+x}\right)\div\left\{1+\frac{(1-x)(1-y)}{(1+x)(1+y)}\right\},$$

and find its value.

The expression

$$=\left\{\frac{1-(a-1)}{1+(a-1)}-\frac{1-(a+1)}{1+(a+1)}\right\}\div\left[1+\frac{\{1-(a+1)\}\{1-(a-1)\}}{\{1+(a+1)\}\{1+(a-1)\}}\right]$$

$$=\left\{\frac{2-a}{a}-\frac{(-a)}{2+a}\right\}\div\left\{1+\frac{(-a)(2-a)}{(2+a)(a)}\right\}$$

$$=\left\{\frac{(2-a)(2+a)+a^2}{a(2+a)}\right\}\div\left\{\frac{a(2+a)-a(2-a)}{a(2+a)}\right\}$$

$$=\frac{4-a^2+a^2}{a(2+a)}\div\frac{2a+a^2-2a+a^2}{a(2+a)}$$

$$=\frac{4}{a(2+a)}\div\frac{2a^2}{a(2+a)}=\frac{4}{a(2+a)}\times\frac{a(2+a)}{2a^2}$$

$$=\frac{2}{a^2}.$$

The questions in Exercise 16c, page 249, should now be attempted.

Exercise 16a.

Reduce to the lowest terms :

1. $\frac{4x^2y^3}{6xy^4}$.
2. $\frac{9xy^4}{3x^2y}$.
3. $\frac{14a^2b^2}{21ab^3}$.
4. $\frac{2l^3n}{ln^2}$.
5. $\frac{a^4b^5}{2a^2b^3}$.
6. $\frac{10x^2y^4}{12x^2y^2}$.
7. $\frac{18a^4b}{6ab}$.
8. $\frac{8c^3d}{2c^4d}$.
9. $\frac{7m^3n}{m^2n}$.
10. $\frac{2x^3yz^2}{3x^4y^2z^3}$.
11. $\frac{5a^3bc^4}{15a^2c}$.
12. $\frac{39p^3q^6}{13p^5q^7}$.
13. $\frac{42m^5n^3}{6lm^3n}$.
14. $\frac{27x^4y^2z^3}{3x^3yz^2}$.
15. $\frac{19a^4b^4c}{57ab^2c^2}$.
16. $\frac{24a^2x^3y^5}{8a^3xy^3}$.
17. $\frac{13p^2q^3}{11p^2q^2}$.
18. $\frac{22c^3d^5}{33b^3c^2d^6}$.
19. $\frac{35a^4b^3c}{42a^7b^2c^3}$.
20. $\frac{3x^2yz^3}{48x^2y^3z^3}$.
21. $\frac{15m^2nx^5}{40mny^5}$.
22. $\frac{10cx}{5acx+15cx}$.
23. $\frac{a^2x^2}{2ax^3-a^2x^2}$.
24. $\frac{a^2+ab}{b^2+ab}$.

25. $\dfrac{a^2-2ab}{ab-2b^2}$.
26. $\dfrac{a^2+9a}{3a^3+27a^2}$.
27. $\dfrac{x^3-x^2y}{xy^2-y^3}$.

28. $\dfrac{4a^3-2a}{6a^2-3}$.
29. $\dfrac{ax^2-3bx^2}{2ax-6bx}$.
30. $\dfrac{4ax-3ay}{3ax-4ay}$.

31. $\dfrac{bc^2d-bd^3}{axc^2-axd^2}$.
32. $\dfrac{m^2n-ln}{m^3nl-mnl^2}$.
33. $\dfrac{2a^2-2}{ma+m}$.

34. $\dfrac{8x^2-2y^2}{2x^2-xy}$.
35. $\dfrac{x^3-3x^2y}{x^3-9xy^2}$.
36. $\dfrac{(2a+3b)^2}{4a^2-9b^2}$.

37. $\dfrac{8c^2-2}{2bc+b}$.
38. $\dfrac{5a^2b+10ab}{a^2b+ab-2b}$.
39. $\dfrac{x^2y^2-1}{2xy-2}$.

40. $\dfrac{c^2+2c-15}{c^2+8c-33}$.
41. $\dfrac{3m+4}{12m^2+19m+4}$.
42. $\dfrac{9x^2-4}{12x^2+5x-2}$.

43. $\dfrac{15a^2+46ab+35b^2}{6a+10b}$.
44. $\dfrac{x^2+2ax+a^2}{3x^2+4ax+a^2}$.
45. $\dfrac{m^2n^2-3mn+2}{4m^2n-8m}$.

46. $\dfrac{c^3+d^3}{c^2-d^2}$.
47. $\dfrac{a^2-2ab+b^2}{a^3-b^3}$.
48. $\dfrac{2x^3-16}{13x^2-2x-48}$.

49. $\dfrac{ab^3+ac^3}{b^2-bc+c^2}$.
50. $\dfrac{4a^2x^2+2ax+1}{16a^3x^3-2}$.
51. $\dfrac{m^4-4n^4}{2m^2n-4n^3}$.

52. $\dfrac{6x^4-13x^2-5}{9x^4-1}$.
53. $\dfrac{p^3-q^3}{q^3-p^3}$.
54. $\dfrac{4l^2-1}{2-2l-12l^2}$.

55. $\dfrac{8a^3+2a-2}{2a^2+3a-2}$.
56. $\dfrac{5x^2+14x-3}{x^3+3x^2-x-3}$.
57. $\dfrac{x^3-3x^2+x-3}{x^3-3x^2-2x+6}$.

58. $\dfrac{a^4+a^2b^2+b^4}{a^3-b^3}$.
59. $\dfrac{a^6+2a^3x^3+x^6}{a^6-x^6}$.

Exercise 16b

Simplify:

1. $\dfrac{ab^2}{x^2y^2}\times\dfrac{x^2y}{b^2}$.
2. $\dfrac{2x^3y}{c^3d}\times\dfrac{c^2d^3}{6xy^3}$.
3. $\dfrac{3mn}{2p^4q}\times\dfrac{4p^3q^3}{6m^2n^3}$.

4. $\dfrac{6x^3y^2}{9a^2b^3}\times\dfrac{3a^3b^2}{2x^2y^3}$.
5. $\dfrac{4m^3}{10cd}\times\dfrac{5c^2}{m^3n}$.
6. $\dfrac{6a^3b}{14p^4q}\times\dfrac{aq}{a^2b}$.

7. $\dfrac{3ax^2}{b^2y}\times\dfrac{2ab}{6x^2y}$.
8. $\dfrac{12m^4n^2}{9n^3b}\times\dfrac{3nb^2}{2bm^4}$.

9. $\frac{7ax^2}{3by^2} \div \frac{14a^2x}{7by}$.

10. $\frac{13d^3c^4}{11a^5b^3} \div \frac{39dc^2}{3a^3b^3}$.

11. $\frac{3ab}{2xy^2} \times \frac{5a^2x}{3b^2y} \div \frac{10a^3}{y^3}$.

12. $\left(\frac{21d^5c}{4a^2b^5} \div \frac{7d^2c^2}{3db}\right) \times \frac{2ab^2}{d^2}$.

13. $\frac{ax^2}{by} \times \frac{bx^3y}{a^2x^4} \div \frac{abx^3}{y}$.

14. $\frac{c^2}{d^2} \times \left(\frac{bc^2}{a^2d} \div \frac{c^3b^2}{a^3d^3}\right) \times \frac{b^2}{d^3}$.

15. $\left(\frac{ax^3}{cy} \times \frac{bx^2}{dy^2}\right) \div \left(\frac{abx}{cdy} \div \frac{y^2}{x}\right)$.

16. $\frac{a}{x+y} \times \frac{x^2+xy}{a^3}$.

17. $\frac{3x}{x-y} \times \frac{x^2y-xy^2}{3x^2}$.

18. $\frac{6c^3}{2a-3b} \times \frac{4a^2-9b}{4ac+6bc}$.

19. $\frac{a^3-a^2}{a^2-1} \times \frac{ax+x}{a^2}$.

20. $\frac{x^2-25}{a^2-b^2} \times \frac{a-b}{x+5}$.

21. $\frac{2x^2-8}{c^2-4d^2} \div \frac{x-2}{c+2d}$.

22. $\frac{1-9a^2}{6a+2} \div \frac{x^3+x^2}{2x^2-2}$.

23. $\frac{c^2d^2-16}{25a^3-a} \times \frac{5a^2+a}{2cd+8}$.

24. $\frac{b-a}{c-d} \times \frac{d^2-c^2}{b^2-a^2}$.

25. $\frac{m^2-n}{2a-3b} \div \frac{m^2n-n^2}{6b-4a}$.

26. $\frac{21-3x}{x+7} \div \frac{x^2-49}{(x+7)^2}$.

27. $\frac{a^2+2ab+b^2}{a^3+b^3} \times \frac{1}{a+b}$.

28. $\frac{a^2-b^2}{ab+b^2} \div \frac{a^2-2ab+b^2}{a^2-ab}$.

29. $\frac{x^3-y^3}{x^2-y^2} \times \frac{x^2y+y^2x}{xy}$.

30. $\frac{x^2+4x+3}{x^2-5x+6} \times \frac{x^2-4x+4}{x^2-1}$.

31. $\frac{c^2+c}{c^2-7c+12} \div \frac{c^2+10c+9}{2c^2-6c}$.

32. $\frac{x^2+10x+16}{x^2+3x-40} \div \frac{x^3+16x^2+28x}{x^2+13x-14}$.

33. $\frac{y^2-2y-8}{y^2-y-6} \times \frac{y^2-7y+12}{y^2-16}$.

34. $\frac{b^2-4b-5}{b^2-25} \div \frac{b^3+1}{b^2+4b-5}$.

35. $\frac{x^2-x-12}{3x^2-27} \times \frac{x^2-8x+15}{x^2-9x+20}$.

36. $\frac{4a^2+3a-1}{3a^2+5a+2} \div \frac{4a^2+4a+1}{6a^2+7a+2}$.

37. $\frac{27-y^3}{6+7y-3y^2} \times \frac{3y^2-4y-4}{y^2-4}$.

38. $\frac{6x^2+x-12}{4+x-3x^2} \times \frac{x^2+3x+2}{2x^2+7x+6}$.

39. $\frac{8c^3+1}{10c^2+3c-1} \div \frac{4c^2-2c+1}{25c^2-1}$.

40. $\frac{2-15x+7x^2}{4-8x+3x^2} \div \frac{2x-14x^2}{4-9x^2}$.

41. $\dfrac{y^4-125y}{y^2+5y+25}\times\dfrac{4y-2}{2y^2-11y+5}$.

42. $\dfrac{6-19d+15d^2}{5d^2-33d+18}\times\dfrac{36-d^2}{d^2+6d}$.

43. $\dfrac{5x+1}{6x^2+x-1}\times\dfrac{9x^2-1}{3x^2-14x-5}\times\dfrac{x^2-2x-15}{5x^2+16x+3}$.

44. $\dfrac{14x^2+19x-3}{5x^2-7x+2}\times\dfrac{25x^2-4}{7x-1}\times\dfrac{x-x^2}{(2x+3)(5x+2)}$.

45. $\dfrac{a^4-b^4}{2a^2-ab-b^2}\times\dfrac{8a^3+b^3}{a^2+b^2}\div\dfrac{a+b}{2}$.

46. $\dfrac{x^3-8y^3}{x^2+2xy-3y^2}\times\dfrac{2x^2+5xy-3y^2}{x^2+2xy+4y^2}\div\dfrac{2x^2-5xy+2y^2}{4x^2-3xy-y^2}$.

47. $\dfrac{2a^2-7ab+3b^2}{a^2-3ab+9b^2}\div\left(\dfrac{6a^2+ab-2b^2}{2a^3+54b^3}\div\dfrac{3a^2+2ab}{a^2-9b^2}\right)$.

48. $\dfrac{c^3+2c^2d+cd^2}{27c^3+64d^3}\times\dfrac{6c^2+5cd-4d^2}{3c^2+6cd+3d^2}\div\dfrac{2c^2-3cd+d^2}{9c^2-12cd+16d^2}$.

49. $\dfrac{4m^2-9n^2}{3m^2n^2}\div\left(\dfrac{4m-6n}{mn}\times\dfrac{2m+3n}{6m^2}\right)$.

50. $\dfrac{16a^4-1}{2a^2-a}\div\left(\dfrac{4a^3+a}{a^2}\times\dfrac{2a+1}{4a}\right)$.

51. $\dfrac{x^4+x^2y^2+y^4}{x^3+y^3}\times\dfrac{x+y}{x^3-y^3}$.

52. $\dfrac{(y+z)^2-x^2}{y^2+yz+yx}\times\dfrac{(z+x)^2-y^2}{(x+y)^2-z^2}\times\dfrac{yx+y^2-yz}{y+z-x}$.

53. $\left(\dfrac{a^2+2ab+b^2-c^2}{a^2+b^2-c^2-2ab}\div\dfrac{a+b+c}{a-b+c}\right)\times\dfrac{(a-c)^2-b^2}{a+b-c}$.

54. $\dfrac{(x+2y)^2}{4x^2-2xy+y^2}\times\dfrac{2x^2-3xy-2y^2}{3x^2-12y^2}\div\dfrac{(2x+y)^2}{8x^3+y^3}$.

55. $\dfrac{ab^2-ad^2}{b^3+d^3}\times\dfrac{ab^2-abd+ad^2}{a^3-a}\times\dfrac{ab-b+ad-d}{ab}$.

56. $\dfrac{9-(x+3)^2}{x^2-2xy+2y^2}\times\dfrac{x^4+4y^4}{x^3+6x^2}\times\dfrac{x-xy}{x^2+2xy+2y^2}$.

57. $\left(\dfrac{3x^2y^2+16xy+5}{27x^3y^3+1}\div\dfrac{x^3y+5x}{x^4}\right)\times\dfrac{9x^2y^2-3xy+1}{x^2y}$.

Exercise 16c

Simplify:

1. $\dfrac{1}{x-1}-\dfrac{1}{x+1}-\dfrac{2}{x^2}$.

2. $\dfrac{2}{x+2}+\dfrac{3}{x-3}-\dfrac{5}{x}$.

3. $\dfrac{1}{x-1}-\dfrac{1}{x-2}-\dfrac{1}{3x-2}$.

4. $\dfrac{2}{x+3}+\dfrac{3}{x-2}-\dfrac{5}{x+1}$.

5. $\dfrac{1}{x+1}+\dfrac{1}{x-1}-\dfrac{1}{x^2-1}$.

6. $\dfrac{2}{x-1}+\dfrac{3}{x+1}+\dfrac{5x}{1-x^2}$.

7. $\dfrac{x}{x-1}-\dfrac{x-1}{x-2}-\dfrac{2}{3-x}$.

8. $\dfrac{3}{2x+3}-\dfrac{4}{3x-1}-\dfrac{1}{6x-3}$.

9. $\dfrac{3x-1}{3x+1}+\dfrac{x+2}{x-3}-\dfrac{4x+3}{2x-1}$

10. $\dfrac{2x-1}{x-3}-\dfrac{x+4}{x-1}-\dfrac{x-3}{x-4}$.

11. $\dfrac{x+2}{x+1}+\dfrac{x-1}{x+2}-\dfrac{2x}{x-1}$.

12. $\dfrac{4x+3}{2x+1}-\dfrac{3x+1}{x-2}+\dfrac{x-3}{x-4}$.

13. $\dfrac{5}{2x^2+x-1}-\dfrac{2}{x^2-x-2}$.

14. $\dfrac{6}{8-2x-x^2}+\dfrac{7}{x^2+x-12}$.

15. $\dfrac{1}{x^2+5x+6}-\dfrac{6}{8+2x-x^2}$.

16. $\dfrac{1}{x^2-2xy+y^2}+\dfrac{1}{x^2-y^2}$.

17. $\dfrac{2a}{a^2-x^2}+\dfrac{1}{x-a}$.

18. $\dfrac{x}{x+1}-\dfrac{1}{x(x-1)}-\dfrac{1}{x(x+1)}$.

19. $\dfrac{2a-b}{a-b}+\dfrac{2b^2}{b^2-a^2}$.

20. $\dfrac{4x+3}{x^2-3x+2}+\dfrac{3(x+1)}{x^2-x-2}$.

21. $\dfrac{2x^2-x}{2x^2+5x-3}+\dfrac{x}{x^2-7x+12}+\dfrac{x}{4-x}$.

22. $\dfrac{12x-25}{8(2x^2-9x+10)}+\dfrac{1}{5-2x}+\dfrac{1}{4(2-x)}-\dfrac{1}{(10-4x)^2}$.

23. $\dfrac{6x^2-12x+4}{\dfrac{1}{x}+\dfrac{1}{x-1}+\dfrac{1}{x-2}}$.

24. $\left(1-\dfrac{2ab}{a^2+b^2}\right)\div\left(\dfrac{a^3-b^3}{a-b}-3ab\right)$.

25. $\dfrac{1}{x-1}-\dfrac{1}{2(x+1)}-\dfrac{x+3}{2(x^2+1)}+\dfrac{4}{1-x^4}$.

CHAPTER XVII

The Remainder Theorem

Functional Notation

119. Expressions such as $3x^2-7x+4$ or $ax^3-2cx^2+3bx-8$, in which no terms contain square roots or other roots, and all the powers of the unknown are positive integers, are called **rational integral functions** of the unknown.

In this chapter we shall deal with rational integral functions only. A function of x is often denoted briefly by a symbol such as $f(x)$ or $F(x)$. (See § 75.)

Thus we may condense our work by saying:

let $\qquad f(x)=3x^2-7x+4,$

or, let $\qquad F(x)=ax^3-2cx^2+3bx-8.$

Then $f(y)$ means that y is written in the place of x in the function by which $f(x)$ has been defined.

Thus, $\qquad f(y)=3y^2-7y+4.$

Similarly, $\qquad f(2)=3(2)^2-7(2)+4,$

and $\qquad F(-3)=a(-3)^3-2c(-3)^2+3b(-3)-8.$

The Factor Theorem

120. We know that the product of a series of numbers will be zero if one of these numbers is zero, no matter what the other numbers may be, e.g. $(5)\times(-7)\times(0)\times(6)=0$.

Conversely, if the product of a series of numbers is zero, then at least one of these numbers must be zero.

Similarly, the product $(x-3)(3x-5)(2x+3)$ cannot be zero unless at least one of the factors be zero.

If the first factor $x-3$ is zero, $x=3$.

If the second factor $3x-5$ is zero, $x=\frac{5}{3}$.

If the third factor $2x+3$ is zero, $x=-\frac{3}{2}$.

Now $(x-3)(3x-5)(2x+3)=6x^3-19x^2-12x+45$.

Hence we learn that the expression $6x^3-19x^2-12x+45$ becomes zero if x be given any of the values 3, $\frac{5}{3}$ or $-\frac{3}{2}$, but for no other values of x.

Conversely, if the substitution $x=3$, that is $(x-3)=0$, makes the expression $6x^3-19x^2-12x+45$ equal to zero, then we may assume that the expression contains the factor $(x-3)$.

Now substituting $x=3$,

$$\begin{aligned}6x^3-19x^2-12x+45&=6(3)^3-19(3)^2-12(3)+45\\&=6(27)-19(9)-12(3)+45\\&=162-171-36+45=0.\end{aligned}$$

Thus we conclude that $(x-3)$ is a factor. Similarly, if $6x^3-19x^2-12x+45$ is also zero for the substitutions $x=\frac{5}{3}$ and $x=-\frac{3}{2}$, then it contains the additional factors $(3x-5)$ and $(2x+3)$.

Generally, **if an expression is zero when a substitution $x=n$ (say) is made, the expression must contain a factor which is zero when $x=n$, that is, it contains the factor $(x-n)$.** This result is usually known as the **Factor Theorem.**

A proof of this result is given later in § 122.

121. Consider the expression $x^3-7x^2+14x-8$.

If we substitute $x=1$, the expression becomes

$$(1)^3-7(1)^2+14(1)-8=1-7+14-8=0.$$

Hence the expression contains the factor $(x-1)$.

If we substitute $x=2$, the expression becomes

$$\begin{aligned}(2)^3-7(2)^2+14(2)-8&=8-7(4)+14(2)-8\\&=8-28+28-8=0.\end{aligned}$$

Hence the expression contains the factor $(x-2)$.

If we substitute $x=4$, the expression becomes

$$\begin{aligned}(4)^3-7(4)^2+14(4)-8&=64-7(16)+14(4)-8\\&=64-112+56-8=0.\end{aligned}$$

Hence the expression contains the factor $(x-4)$.

When the three factors obtained are multiplied out, the first term is x^3. Hence the original cubic expression contains no other factors.

Thus, $\quad x^3-7x^2+14x-8=(x-1)(x-2)(x-4)$.

Example 1. Show that $(x-2)$, $(x-5)$, $(x+3)$ and $(x+4)$ are factors of $x^4-27x^2-14x+120$.

For brevity we shall denote the expression $x^4-27x^2-14x+120$ by $f(x)$.

Now $\quad f(2)=(2)^4-27(2)^2-14(2)+120=16-27(4)-14(2)+120$
$$=16-108-28+120=0.$$

Hence $f(x)$ contains the factor $(x-2)$.

Also, $f(5)=(5)^4-27(5)^2-14(5)+120=625-27(25)-14(5)+120$
$$=625-675-70+120=0.$$

Hence $f(x)$ contains the factor $(x-5)$.

Also,
$$f(-3)=(-3)^4-27(-3)^2-14(-3)+120=81-27(9)+14(3)+120$$
$$=81-243+42+120=0.$$

Hence $f(x)$ contains the factor $[x-(-3)]$, that is the factor $(x+3)$.

Also
$$f(-4)=(-4)^4-27(-4)^2-14(-4)+120=256-27(16)+14(4)+120$$
$$=256-432+56+120=0.$$

Hence $f(x)$ contains the factor $[x-(-4)]$, that is the factor $(x+4)$.

The questions in Exercise 17a, page 262, should now be attempted.

Proof of Remainder and Factor Theorems

122. When a function of x, $F(x)$ say, is divided by the linear expression $ax+b$, the division can be continued until a remainder, R say, is obtained which does not contain x.

If the quotient obtained by this division is $Q(x)$, then
$$F(x)=(ax+b)Q(x)+R. \quad . \quad . \quad . \quad (1)$$

[Compare in arithmetic:

$$\begin{array}{l} 7)25(3 \\ \;\;\underline{21} \\ \;\;\;\;\underline{4} \end{array} \qquad \therefore \quad 25=7\times3+4.]$$

The statement in line (1) is true whatever value x may have. If we give x the value which makes $ax+b=0$, that is, if $x=-\frac{b}{a}$, line (1) becomes:
$$F\left(-\frac{b}{a}\right)=0\times Q\left(-\frac{b}{a}\right)+R \quad . \quad . \quad . \quad (2)$$

The R in line (2) is the same as in line (1), since it does not contain x.

Thus,
$$F\left(-\frac{b}{a}\right)=R.$$

Thus, if $F(x)$ is divided by $ax+b$ until a remainder is obtained which does not contain x, then the remainder is equal to $F\left(-\frac{b}{a}\right)$, i.e. to the value of the function when $ax+b=0$.

This result is known as the **Remainder Theorem.**

If it is found that $F\left(-\frac{b}{a}\right)=0$, i.e. that the remainder obtained when $F(x)$ is divided by $ax+b$ is zero, **then $ax+b$ is a factor of $F(x)$.** This is another statement of the **Factor Theorem.**

By putting $a=1$, and $b=-n$, in the above results we obtain:

(1) When $F(x)$ is divided by $x-n$ the remainder is $F(n)$.

(2) If $F(n)=0$, then $x-n$ is a factor of $F(x)$.

Example 2. Without division find the remainder when $2x^3-5x^2+2x+5$ is divided by $x-3$.

By the remainder theorem, the remainder is the value the expression assumes when $x-3=0$, i.e. when $x=3$.

Thus, the remainder$=2(3)^3-5(3)^2+2(3)+5$

$$=2(27)-5(9)+2(3)+5$$

$$=54-45+6+5=20.$$

The pupil should verify this result by actual division.

Example 3. What number must be subtracted from $7x^3-9x^2-8x+6$ to make it divisible by $x-2$ without remainder?

Denote the expression by $f(x)$. Then the remainder when $f(x)$ is divided by $x-2$ is $f(2)$.

$$f(2)=7(2)^3-9(2)^2-8(2)+6$$

$$=7(8)-9(4)-8(2)+6$$

$$=56-36-16+6=10.$$

Thus 10 must be subtracted from $f(x)$ to leave no remainder on division by $(x-2)$.

The pupil should verify that $(x-2)$ divides exactly into $f(x)-10$, that is into $7x^3-9x^2-8x-4$.

The questions in Exercise 17b, page 263, should now be attempted.

Choosing the Substitutions

123. Example 4. Factorise x^3-4x^2+x+6.

Denote the expression x^3-4x^2+x+6 by $f(x)$.

If $f(x)$ can be factorised, then the product of the constant terms in the factors gives the constant term in $f(x)$. Hence **the constant terms in the factors to be found must be factors of the constant term in $f(x)$,** which in this example is 6.

Thus the only substitutions we need try are:

$$x=\pm1,\ \pm2,\ \pm3, \text{ or } \pm6.$$

$$f(1)=(1)^3-4(1)^2+(1)+6=4.$$

Since $f(1)$ is not zero, $(x-1)$ is not a factor of $f(x)$.

$$f(-1)=(-1)^3-4(-1)^2+(-1)+6=0.$$

Since $f(-1)$ is zero, $(x+1)$ is a factor of $f(x)$.

$$f(2)=(2)^3-4(2)^2+(2)+6=0.$$

Since $f(2)$ is zero, $(x-2)$ is a factor of $f(x)$.

$$f(-2)=(-2)^3-4(-2)^2+(-2)+6=-20.$$

Since $f(-2)$ is not zero, $(x+2)$ is not a factor of $f(x)$.

$$f(3)=(3)^3-4(3)^2+(3)+6=0.$$

Since $f(3)$ is zero, $(x-3)$ is a factor of $f(x)$.

Since $f(x)$ is of the third degree, and we have found three factors, there are no further factors.

Thus, $x^3-4x^2+x+6=(x+1)(x-2)(x-3)$.

Note.—The degree of an expression is not altered by a numerical factor. Hence in addition to the three factors found, $f(x)$ might have a numerical factor. It will be seen, however, that as the product of the three factors commences with the term x^3, and $f(x)$ also commences with the term x^3, the numerical factor is unity.

124. Example 5. Factorise $2x^3-3x^2-2x+3$.

Denote the expression by $f(x)$.

If $f(x)$ can be factorised, then the product of the coefficients of x in the factors gives the coefficient of the highest power of x in $f(x)$.

Hence **the coefficients of x in the factors to be found must be factors of the coefficient of the highest power of x in $f(x)$**, which in this example is 2.

Thus the factors must have either x or $2x$ as their first terms.

The constant terms in the factors must be ± 1 or ± 3.

Thus the only substitutions we need try are

$$\pm 1, \pm 3, \pm \tfrac{1}{2}, \pm \tfrac{3}{2}.$$

$$f(\tfrac{1}{2})=2(\tfrac{1}{2})^3-3(\tfrac{1}{2})^2-2(\tfrac{1}{2})+3=1\tfrac{1}{2}.$$

Since $f(\frac{1}{2})$ is not zero, $(2x-1)$ is not a factor.

$$f(-\tfrac{1}{2})=2(-\tfrac{1}{2})^3-3(-\tfrac{1}{2})^2-2(-\tfrac{1}{2})+3=3.$$

Since $f(-\frac{1}{2})$ is not zero, $(2x+1)$ is not a factor.

$$f(1)=2(1)^3-3(1)^2-2(1)+3=0.$$

Since $f(1)$ is zero, $(x-1)$ is a factor.

$$f(-1)=2(-1)^3-3(-1)^2-2(-1)+3=0.$$

Since $f(-1)$ is zero, $(x+1)$ is a factor.

$$f(\tfrac{3}{2})=2(\tfrac{3}{2})^3-3(\tfrac{3}{2})^2-2(\tfrac{3}{2})+3=0.$$

Since $f(\frac{3}{2})$ is zero, $(2x-3)$ is a factor.

Since $f(x)$ is of the third degree and we have found three factors, there are no further factors.

Hence $\quad 2x^3-3x^2-2x+3=(x-1)(x+1)(2x-3).$

Note.—As explained in the previous example, there may be an additional numerical factor. It will be seen, however, that as the product of the three factors found begins with the term $x\times x\times 2x$, that is $2x^3$, and $f(x)$ begins with the term $2x^3$, the numerical factor is unity.

The Final Factor

125. Example 6. Factorise $\quad 6x^4-23x^3-45x^2+194x-120.$

Denote the expression by $f(x)$.

$$f(1)=6(1)^4-23(1)^3-45(1)^2+194(1)-120=12.$$
$$f(-1)=6(-1)^4-23(-1)^3-45(-1)^2+194(-1)-120=-330.$$
$$f(2)=6(2)^4-23(2)^3-45(2)^2+194(2)-120=0.$$
$$f(-2)=6(-2)^4-23(-2)^3-45(-2)^2+194(-2)-120=-408.$$
$$f(3)=6(3)^4-23(3)^3-45(3)^2+194(3)-120=-78.$$
$$f(-3)=6(-3)^4-23(-3)^3-45(-3)^2+194(-3)-120=0.$$
$$f(4)=6(4)^4-23(4)^3-45(4)^2+194(4)-120=0.$$

Hence three factors of $f(x)$ are $(x-2)$, $(x+3)$, $(x-4)$.

As $f(x)$ is of the fourth degree, it contains one more linear factor. The first term of this factor when multiplied by $x \times x \times x$ must give the first term of $f(x)$, which is $6x^4$. Hence the required first term is $6x$. The constant term of this factor when multiplied by $(-2)\times(3)\times(-4)$ must give the constant term in $f(x)$, which is -120. Hence the required constant term is -5. Thus the required factor is $(6x-5)$. This can be verified by making the substitution $x=\frac{5}{6}$ in $f(x)$. The student should verify that $f(\frac{5}{6})=0$.

Thus, $6x^4-23x^3-45x^2+194x-120=(x-2)(x+3)(x-4)(6x-5)$.

Note.—When all the factors of an expression, with the exception of one linear factor, have been found, that linear factor can always be obtained by the method adopted in the above example.

126. **Example 7.** Factorise $6x^5-19x^4-36x^3+111x^2+58x-120$.

Denote the expression by $f(x)$.

$$f(1)=6(1)^5-19(1)^4-36(1)^3+111(1)^2+58(1)-120 \qquad =0.$$

$$f(-1)=6(-1)^5-19(-1)^4-36(-1)^3+111(-1)^2+58(-1)-120=-56.$$

$$f(2)=6(2)^5-19(2)^4-36(2)^3+111(2)^2+58(2)+120 \qquad =+40.$$

$$f(-2)=6(-2)^5-19(-2)^4-36(-2)^3+111(-2)^2+58(-2)-120=0.$$

$$f(3)=6(3)^5-19(3)^4-36(3)^3+111(3)^2+58(3)-120 \qquad =0.$$

Hence three factors of $f(x)$ are $(x-1)$, $(x+2)$ and $(x-3)$.

If the product of these factors be divided into $f(x)$, the quotient will contain the remaining factors of the expression.

It will be found that $(x-1)(x+2)(x-3)=x^3-2x^2-5x+6$, and that $f(x)\div(x^3-2x^2-5x+6)=6x^2-7x-20$.

The factors of $6x^2-7x-20$ are given by $\begin{array}{c|c} 3 & 4 \\ \hline 2 & -5 \end{array}$ and are thus $(3x+4)$ and $(2x-5)$.

Thus, $f(x)=(x-1)(x+2)(x-3)(3x+4)(2x-5)$.

Note.—When all but two of the factors of a given expression have been found by the factor theorem, the remaining two can be obtained by dividing the expression by the product of those found, and factorising the resulting quadratic.

This procedure is useful when, as in the above example, two of the factors would involve fractional substitutions for x. It is also necessary when an expression contains a repeated factor

such as $(x-3)^2$, since the application of the factor theorem will show that the expression contains the factor $(x-3)$, but cannot show that it contains the factor a second time.

The questions in Exercise 17c, page 264, should now be attempted.

Contracted Method for Substitution

127. Show that $(x-3)$, $(x-5)$, $(x+5)$ and $(x+6)$ are factors of the expression $x^4+3x^3-43x^2-75x+450$.

Making the substitution $x=3$, let us first eliminate x^4.

Now $x^4=x\times x^3$. $\therefore$ when $x=3$, $x^4=3x^3$.

$$\begin{aligned}
\therefore\quad x^4+3x^3-43x^2-75x+450 &= 6x^3-43x^2-75x+450.\\
(\text{When } x=3,\ 6x^3=18x^2) &= -25x^2-75x+450.\\
(\text{When } x=3,\ -25x^2=-75x) &= -150x+450.\\
(\text{When } x=3,\ -150x=-450) &= 0.
\end{aligned}$$

Thus when $x=3$ the expression$=0$.

Hence $(x-3)$ is a factor of the expression.

This method of working, which avoids a great deal of multiplication, may be set out as follows:

x^4	x^3	x^2	x		
1	3	-43	-75	450	. . (A)
	3	18	-75	-450	. . (B)
	6	-25	-150	0	. . (C)

We make a column for each power of x, and one for the constants. In line (A) we write down the coefficients of the terms in the given expression.

$x^4=3x^3$. We therefore add 3 (line B) to the coefficient of x^3, making a total of 6 (line C).

$6x^3=18x^2$. We therefore add 18 (line B) to the coefficient of x^2, making a total of -25 (line C).

$-25x^2=-75x$. We therefore add -75 (line B) to the coefficient of x, making a total of -150 (line C).

$-150x=-450$. We therefore add -450 (line B) to the constant term, making a total of 0 (line C).

After some practice the headings of the columns may be omitted.

Making the substitution $x=5$.

Now $x^4=x\times x^3$ $\therefore$ when $x=5$, $x^4=5x^3$, etc.

The working is set out as follows:

1	3	-43	-75	450
	5	40	-15	-450
	8	-3	-90	0

Thus the value of the expression when $x=5$ is 0, so that $(x-5)$ is a factor of the expression.

Making the substitution $x=-5$:

1	3	-43	-75	450
	-5	10	165	-450
	-2	-33	90	0

Thus the remainder is 0, and $(x+5)$ is a factor.

Making the substitution $x=-6$:

1	3	-43	-75	450
	-6	18	150	-450
	-3	-25	75	0

Thus the remainder is 0, and $(x+6)$ is a factor.

Hence $x^4+3x^3-43x^2-75x+450$ contains the factors

$$(x-3),\ (x-5),\ (x+5),\ (x+6).$$

Example 8. Factorise $x^5-20x^3-30x^2+19x+30$.

The pupil should notice that as there is no term in x^4, it is necessary to insert 0 in the appropriate place.

Making the substitution $x=1$:

1	0	-20	-30	19	30
	1	1	-19	-49	-30
	1	-19	-49	-30	0

Thus the remainder is 0, and $(x-1)$ is a factor.

Making the substitution $x=-1$:

1	0	-20	-30	19	30
	-1	1	19	11	-30
	-1	-19	-11	30	0

Thus the remainder is 0, and $(x+1)$ is a factor.

Making the substitution $x=2$:

1	0	-20	-30	19	30
	2	4	-32	-124	-210
	2	-16	-62	-105	-180

Thus the remainder is -180, so that $(x-2)$ is not a factor.

Making the substitution $x=-2$:

1	0	-20	-30	19	30
	-2	4	32	-4	-30
	-2	-16	2	15	0

Thus the remainder is 0, so that $(x+2)$ is a factor.

Making the substitution $x=3$:

1	0	-20	-30	19	30
	3	9	-33	-189	-510
	3	-11	-63	-170	-480

Thus the remainder is -480, so that $(x-3)$ is not a factor.

Making the substitution $x=-3$:

1	0	-20	-30	19	30
	-3	9	33	-9	-30
	-3	-11	3	10	0

Thus the remainder is 0, and $(x+3)$ is a factor.

The expression is of the fifth degree, and four factors have been found. Therefore the final factor has a constant term:

$$\frac{30}{(-1)(1)(2)(3)}=-5. \quad \text{(See note, Ex. 6.)}$$

The final factor is thus $(x-5)$.

Hence the expression$=(x-1)(x+1)(x+2)(x+3)(x-5)$.

128. Example 9. Find the values of a and b so that the expression $x^4-x^3+ax^2+x+b$ should be exactly divisible by x^2-3x+2.

Now $x^2-3x+2=(x-1)(x-2)$; hence the given expression must be divisible both by $(x-1)$ and by $(x-2)$. Therefore the substitutions $x=1$ and $x=2$ must both make the expression zero.

Making the substitution $x=1$:

$$(1)^4-(1)^3+a(1)^2+1+b=0,$$

i.e. $$a+b+1=0 \quad . \quad . \quad . \quad (1)$$

Making the substitution $x=2$:

$$(2)^4-(2)^3+a(2)^2+2+b=0,$$

i.e. $$4a+b+10=0 \quad . \quad . \quad . \quad (2)$$

Solving the equations (1) and (2), we obtain the values $a=-3$, $b=2$.

Example 10. Find the values of a and b in the expression $x^4-2x^3+ax^2+bx+1$, if when it is divided by $(x+3)$ the remainder is 22, and when divided by $(x-2)$ the remainder is -23.

Let $$f(x)=x^4-2x^3+ax^2+bx+1,$$

$$\therefore \quad f(-3)=81-2(-27)+9a-3b+1$$
$$=136+9a-3b.$$

Hence $$136+9a-3b=22,$$
$$\therefore \quad 9a-3b=-114,$$
$$\therefore \quad 3a-b=-38 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also $$f(2)=16-2(8)+4a+2b+1$$
$$=4a+2b+1.$$

Hence $$4a+2b+1=-23,$$
$$\therefore \quad 4a+2b=-24,$$
$$\therefore \quad 2a+b=-12 \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Solving equations (1) and (2), we obtain $a=-10$, $b=8$.

129. Divisibility of x^n-y^n and x^n+y^n by $x-y$ and $x+y$.

1. x^n-y^n is divisible by $x-y$ for all integral values of n.

Let $$f(x)=x^n-y^n,$$
$$\therefore \quad f(y)=y^n-y^n=0,$$
$$\therefore \quad (x-y) \text{ is a factor of } x^n-y^n.$$

2. x^n-y^n is divisible by $x+y$ for all even values of n.

Let $$f(x)=x^n-y^n,$$
$$\therefore \quad f(-y)=(-y)^n-y^n$$
$$=y^n-y^n, \text{ if } n \text{ is even},$$
$$=0.$$
$$\therefore \quad (x+y) \text{ is a factor of } x^n-y^n, \text{ if } n \text{ is even}.$$

3. **x^n+y^n is divisible by $x+y$ for all odd values of n.**

Let $$f(x)=x^n+y^n,$$

$$\therefore \quad f(-y)=(-y)^n+y^n$$
$$=-y^n+y^n, \text{ if } n \text{ is odd,}$$
$$=0.$$

$\therefore$ $(x+y)$ is a factor of x^n+y^n, if n is odd.

4. x^n+y^n is never divisible by $(x-y)$

Let $$f(x)=x^n+y^n,$$

$$\therefore \quad f(y)=y^n+y^n=2y^n.$$

Since $f(y)$ is not zero, $(x-y)$ is not a factor of x^n+y^n.

5. x^n+y^n is never divisible by $(x+y)$, when n is even.

Let $$f(x)=x^n+y^n,$$

$$\therefore \quad f(-y)=(-y)^n+y^n$$
$$=y^n+y^n, \text{ if } n \text{ is even,}$$
$$=2y^n.$$

Since $f(-y)$ is not zero, $x+y$ is not a factor of x^n+y^n, when n is even.

The following illustrations of results 1, 2 and 3 should be verified by actual division.

1. $$\frac{x^2-y^2}{x-y}=x+y.$$

$$\frac{x^3-y^3}{x-y}=x^2+xy+y^2.$$

$$\frac{x^4-y^4}{x-y}=x^3+x^2y+xy^2+y^3.$$

2. $$\frac{x^2-y^2}{x+y}=x-y.$$

$$\frac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3.$$

$$\frac{x^6-y^6}{x+y}=x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5.$$

3. $$\frac{x^3+y^3}{x+y}=x^2-xy+y^2.$$

$$\frac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4.$$

$$\frac{x^7+y^7}{x+y}=x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6.$$

The questions in Exercise 17d, page 265, should now be attempted.

Exercise 17a

1. If $f(x)=x^4-3x^2+7$, write down the values of $f(a), f(b), f(y)$.
2. Find the value of $3x^4-x^2+1$, (*a*) when $x=1$, (*b*) when $x=2$.
3. If $f(n)=\frac{n}{2}\left\{n+1\right\}$, find the values of $f(5), f(100)$.
4. If $f(x)=x^4+x^3+x^2+1$, write down the values of $f(-a), f(-y)$.
5. What is the value of $f(-2)$ if $f(x)=ax^2+bx+c$?
6. Find the value of x^3+3x^2+3x+1, (*a*) when $x=-1$, (*b*) when $x=-2$.
7. Find the value of $f(n+1)-f(n)$, if $f(n)=\frac{n}{2}(n+1)$.
8. If $f(n)=\frac{n}{6}(n+1)(2n+1)$, find the values of $f(20), f(-10), f(0)$.
9. If $f(x)=(x+1)(x-2)$, show that $f(-1)=0$, and $f(2)=0$.
10. If $F(x)=x^3-3ax^2+3a^2x-a^3$, find the values of $f(a)$, $f(-a)$, $f(-x)$.
11. If $f(x)=x(x+1)(x-1)$, find the values of $f(0)$, $f(-1)$, $f(x+1)-f(x-1)$.
12. Find the value of $12y^2-y-6$, (*a*) when $y=\frac{3}{4}$, (*b*) when $y=-\frac{2}{3}$.
13. If $3x^2+4ax+a^2=f(x)$, find the values of $f\left(-\frac{a}{3}\right), f(2), f(-a)$.
14. If $f(n)=\frac{n}{6}(n+1)(n+2)$, find the value of $f(n)-f(n-1)$; hence find the value of $f(100)-f(99)$.
15. Show that if $f(x)=x^3-6x^2+11x-6$, then $f(1)$, $f(2)$ and $f(3)$ are each 0.
16. If $f(x)=(2x-1)(x-3)(x-2)$, show that $f(\frac{1}{2}), f(3), f(2)$ are each equal to 0. What do you infer ?

17. Show that $f(-1)$, $f(2)$, $f(-\frac{1}{2})$ are each 0 if $f(x)=(x+1)(x-2)(2x+1)$. What do you infer?
18. If $f(x)=6x^3-25x^2+3x+4$, show that $f(4), f(-\frac{1}{3}), f(\frac{1}{2})$ are each 0. What do you infer?
19. Show that $(2x+1)$, $(x-3)$, $(2x-3)$ are each factors of $4x^3-16x^2+9x+9$.
20. Show that $2x^3+7x^2+7x+2$ contains the factors $(2x+1)$, $(x+2)$ and $(x+1)$.
21. Show that $(x+2)$ and $(3x-1)$ are not factors of x^3+3x^2+3x+1, but that $(x+1)$ is a factor.
22. Which of the following expressions contains the factor $(x-2)$?
    (a) $3x^3+11x^2+13x+6$.
    (b) $3x^3-5x^2-4$.
    (c) $5x^4-2x^3+x^2-3x+7$.
23. Is $(x-1)$ a factor of $x^{13}-7x+6$?
24. Is $(x+1)$ a factor of $x^{10}+11x^2+5x-7$?
25. Show that $(x-a)$ is a factor of $3x^4-2ax^3-a^2x^2$.
26. Show that $(x+2b)$ is a factor of $2x^3-bx^2-9b^2x+2b^3$.
27. Show that $3x^3-10x^2-27x+10$ contains the factors $(x-5)$, $(x+2)$ and $(3x-1)$. Does the expression contain any other factors? State the reason for your answer.
28. Factorise x^2+5x+6, and show that $3x^4+14x^3+12x^2-11x-6$ contains the factors of x^2+5x+6.
29. Show that $8x^4+2x^3-33x^2-8x+4$ contains a factor of $2x^2-3x-2$.
30. Show that x^6-1 contains the factor x^2-1.

Exercise 17b

Find the remainder:

1. When x^3+3x^2-2x+3 is divided by $x-1$.
2. When $2x^5-3x^2+3x-5$ is divided by $x+2$.
3. When $8x^3-12x^2+6x-3$ is divided by $2x-1$.
4. When $5x^3+7x^2-8x-3$ is divided by $x+1$.
5. When $x^4-3x^2+2x-11$ is divided by $x-3$.
6. When $2x^3-5x+8$ is divided by $2x-3$.
7. When $x^3+3ax^2+3a^2x+a^3$ is divided by $x-a$.
8. When $3x^3-7ax^2+11a^2x+2a^3$ is divided by $3x-a$.
9. When $x^3-2x^2b+3xb^2-8b^3$ is divided by $x-2b$.
10. When $7x^5+3x^3-2x^2-11$ is divided by $x+p$.
11. When x^3+3x^2-4x+2 is divided by $2x+y$.

12. When x^6-a^6 is divided by $x-a$.
13. When x^7+y^7 is divided by $x+y$.
14. When $3x^4-2x^2y^2+5xy^3-y^4$ is divided by $x+2y$.
15. When ax^3-5cx^2-c is divided by $2x+a$.

16. What number must be subtracted from $5x^3-2x^2+11$ to make it exactly divisible by $x-1$?
17. What number must be added to $x^4+6x^3-3x^2-8x$ to make it exactly divisible by $x+2$?
18. Find the value of k in order that $2x^3-6x^2-x+k$ may be exactly divisible by $2x+1$.
19. Find the value of a if ax^2-3x-2 is to be exactly divisible by $3x-2$.
20. Show that $9x^3-43x-14$ is exactly divisible by $3x^2+7x+2$.
21. Show that $x^2+bx-2b^2$ divides into $x^5-3b^2x^3+2b^3x^2$ without remainder.
22. If x^3-3x^2+4 and x^3+x^2-7x+a leave the same remainder when divided by $x-3$, find the value of a.
23. Prove that $a+b$ must be unity if x^5-ax-b is exactly divisible by $x-1$.
24. Prove that x^n+1 is always divisible by $x+1$, provided n is odd, and is never divisible by $x-1$.
25. Prove that x^n-1 is always divisible by $x-1$ and is divisible by $x+1$ when n is an even integer.

Exercise 17c

Factorise :

1. x^3-2x^2-x+2.
2. x^3+3x^2-x-3.
3. $x^3-6x^2+11x-6$.
4. x^3-7x+6.
5. x^3-2x^2-5x+6.
6. $x^3+3x^2-13x-15$.
7. $x^3-5x^2-4x+20$.
8. x^3+4x^2+x-6.
9. $x^3+3x^2-10x-24$.
10. $x^3+7x^2+7x-15$.
11. $2x^3-7x^2-17x+10$.
12. $2x^3-9x^2+7x+6$.
13. $3x^3-x^2-3x+1$.
14. $2x^3-x^2-22x-24$.
15. $10x^3+3x^2-9x-4$.
16. $6x^3+7x^2-16x-12$.
17. $12x^3+67x^2+36x+5$.
18. $2x^4-x^3-29x^2+34x+24$.
19. $6x^4+7x^3-9x^2-7x+3$.
20. $20x^5+3x^4-82x^3-12x^2+8x$.
21. $14x^4+37x^3-23x^2-25x-3$.

Find one linear factor in each of the following, and find the remaining factor by division :

22. $7x^3-10x^2-38x+15$.

23. $3x^3-28x^2-24x+11$.

24. $2x^3+3x^2+8x+12$.

25. $6x^3+24x^2-5x-20$.

26. $x^3-5x+12$.

27. $3x^5-2x^4-6x+4$.

28. $4x^5-8x^4-11x+22$.

29. $2x^4-x^3-4x+2$.

Exercise 17d

Factorise :

1. $x^4+5x^3-13x^2-77x-60$.

2. $6x^4-x^3-83x^2-54x+72$.

3. $6x^4-5x^3-75x^2-10x+24$.

4. $9x^4+9x^3-112x^2-4x+48$.

5. $2x^5+3x^4-10x^3-15x^2+8x+12$.

6. $12x^5+8x^4-165x^3-245x^2+18x+72$.

7. $12x^5-10x^4-108x^3-46x^2+96x+56$.

8. $3x^4-x^3-11x^2+4x-4$.

9. $2x^5-5x^4+2x^3-2x^2+5x-2$.

10. $x^4+3x^3+4x^2+3x+1$.

11. $5x^4-3x^3-12x^2+13x-3$.

12. $2x^5-9x^4+6x^3+23x^2-36x+12$.

13. $x^5+2x^4-x^3-5x^2-4x-1$.

14. $x^3-7xy^2-6y^3$.

15. $x^3-3x^2y+3xy^2-y^3$.

16. $x^3+(2a-1)x^2-(2a-a^2)x-a^2$.

17. $2x^4-7x^3y+6x^2y^2+xy^3-2y^4$.

18. $a^4-3a^3b+4a^2b^2-3ab^3+b^4$.

Use the factor theorem to prove :

19. $(a+b)(b+c)(c+a)\equiv(a+b+c)(bc+ca+ab)-abc$.

20. $-(b-c)(c-a)(a-b)\equiv ab(a-b)+bc(b-c)+ca(c-a)$.

21. If $2x^4-x^3+11x^2+ax+b$ is to be exactly divisible by $2x^2+5x+2$, what values must a and b have ?

22. Find a and b in order that $4x^4+ax^3-31x^2+bx+18$ may be exactly divisible by x^2-x-6.

23. Show that $4ax^2+3a^2x-b^3$ is exactly divisible by $x+a$ when $a^2+ab+b^2=0$.

24. The expressions $3x^3-ax^2-5x+2$ and x^3+x^2+bx-5 leave the same remainder when divided by $x+1$, but when divided by $x-3$, the first expression leaves a remainder greater by 16 than that left by the second. Find a and b.

CHAPTER XVIII

Highest Common Factor

130. The factor of highest degree which divides exactly into a set of expressions is called the highest common factor (H.C.F.) of those expressions (see § 88).

Thus the H.C.F. of $2a^2b^3c$ and $3a^3bc^2$ is a^2bc.

The numerical factor of the H.C.F. is found by obtaining the H.C.F. of all numerical coefficients of the expressions.

Example 1. Find the H.C.F. of $18a^3x^4y^2$, $45ax^5y^3$, $27a^4x^3y^4$.

The H.C.F. of 18, 45 and 27 is 9.

The H.C.F. of $a^3x^4y^2$, ax^5y^3 and $a^4x^3y^4$ is ax^3y^2.

Thus the required H.C.F. is $9ax^3y^2$.

The questions in Exercise 18a, page 273, should now be attempted.

H.C.F. by Factorising

131. If we have to find the H.C.F. of expressions which can be readily factorised, the above process can be applied.

Example 2. Find the H.C.F. of $180x^2+15x-90$, $30x^2-65x+30$ and $90x^2+60x-80$.

The first step is to remove the common numerical factors of the coefficients in each expression, thus :

$$15(12x^2+x-6),\ 5(6x^2-13x+6),\ 10(9x^2+6x-8).$$

The numerical factor of the H.C.F. required is the H.C.F. of 15, 5, 10, which is 5.

The second step is to factorise the remaining portions of the expressions, thus :

$$(3x-2)(4x+3),\ (2x-3)(3x-2),\ (3x-2)(3x+4).$$

The H.C.F. of these is $(3x-2)$.

Hence the H.C.F. of the original expressions is $5(3x-2)$, that is, $15x-10$.

Example 3. Find the H.C.F. of

$$12a^5b^2-12a^2b^5,\ 45a^3b^3+9a^2b^4-54ab^5 \text{ and } 15a^5b^3-15a^3b^5.$$

The expressions may be written

$$12(a^5b^2-a^2b^5),\ 9(5a^3b^3+a^2b^4-6ab^5) \text{ and } 15(a^5b^3-a^3b^5).$$

The numerical factors are 12, 9 and 15, so that the numerical factor of the H.C.F. is 3.

Factorising the remaining portions :

$$a^5b^2-a^2b^5=a^2b^2(a^3-b^3)=a^2b^2(a-b)(a^2+ab+b^2).$$
$$5a^3b^3+a^2b^4-6ab^5=ab^3(5a^2+ab-6b^2)=ab^3(a-b)(5a+6b).$$
$$a^5b^3-a^3b^5=a^3b^3(a^2-b^2)=a^3b^3(a-b)(a+b).$$

The H.C.F. of these is $ab^2(a-b)$.

Thus the H.C.F. of the original expression is $3ab^2(a-b)$, that is $3a^2b^2-3ab^3$.

The questions in Exercise 18b, page 274, should now be attempted.

132. Example 4. Find the H.C.F. of $4x^4+5x^3-4x^2+11x+14$ and $(2x-3)(x+2)(2x+5)$.

The first expression cannot be readily factorised, but as the factors of the second expression are given, the only factors which can be *common* to the two are $(2x-3)$, $(x+2)$ and $(2x+5)$. It will be seen that the first and last of these will not divide exactly into $4x^4+5x^3-4x^2+11x+14$, since 14 is divisible neither by 3 nor 5.

Making the substitution $x=-2$:

4	5	−4	11	14
	−8	6	−4	−14
	−3	2	7	0

We see that the remainder is 0, so that $(x+2)$ is a factor of the first expression. Thus the H.C.F. is $x+2$.

Example 5. Find the H.C.F. of

$$36a^4-6a^3-72a^2 \text{ and } 40a^4-52a^3+4a^2-24a.$$

The first expression $=6a^2(6a^2-a-12)=6a^2(2a-3)(3a+4)$.

The second expression $=4a(10a^3-13a^2+a-6)$.

It will be found that $10a^3-13a^2+a-6$ is exactly divisible by $2a-3$, but not by $3a+4$.

Further, the H.C.F. of $6a^2$ and $4a$ is $2a$.

Hence the H.C.F. of the two given expressions is $2a(2a-3)$, that is $4a^2-6a$.

The question in Exercise 18c, page 274, should now be attempted.

General Method for H.C.F.

133. When none of the given expressions can be readily resolved into factors, it is still possible to find their H.C.F. by applying these two principles:

(1) If an expression contains a certain factor, any multiple of the expression also contains that factor.

Suppose the expression A contains the factor g, then g divides into A exactly. Suppose the resulting quotient is Q.

Then $\quad \dfrac{A}{g}=Q, \quad \therefore\ A=gQ.$

Let m stand for any multiple.

Then $\quad mA=mgQ=g(mQ).$

This shows that g is also a factor of mA.

Thus, in arithmetic, 6 is a factor of 18,

$\therefore$ 6 is a factor of 5×18 and of 13×18.

(2) If two expressions contain a common factor, their sum and their difference and also the sum and difference of any multiples of the expressions contain that factor.

Suppose that the two expressions A and B contain the common factor g. Then g divides exactly into both A and B.

Suppose the resulting quotients are Q and R respectively.

Then $\quad \dfrac{A}{g}=Q, \quad$ and $\quad \dfrac{B}{g}=R.$

$\therefore \quad A=gQ, \quad$ and $\quad B=gR.$

Let m and n stand for any multiples.

Then $\quad mA=mgQ, \quad$ and $\quad nB=ngR.$

$\therefore \quad mA+nB=mgQ+ngR=g(mQ+nR),$

and $\quad mA-nB=mgQ-ngR=g(mQ-nR).$

This shows that g is also a common factor of the sum and the difference of any multiples of A and B.

Thus, in arithmetic, 7 is a common factor of 21 and 35.

$\therefore$ 7 is a common factor of $35+21$ and $35-21$.

Again 7 is a common factor of $8\times35-3\times21$ and $9\times35-11\times21$.

134. Example 6. Find the H.C.F. of $6a^3-15a^2+a+5$ and $12a^3-27a^2-7a+15$.

$$\begin{array}{l}
6a^3-15a^2+a+5)12a^3-27a^2-7a+15(2 \\
\qquad\qquad\qquad\quad \underline{12a^3-30a^2+2a+10} \\
\qquad\qquad\qquad\qquad\quad 3a^2-9a+5)6a^3-15a^2+a+5(2a+1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; \underline{6a^3-18a^2+10a} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3a^2-9a+5 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 3a^2-9a+5
\end{array}$$

The required H.C.F. is $3a^2-9a+5$.

Explanation.—The given expressions are arranged in descending powers of a. If the first terms of the two expressions are of different degree, we commence by dividing the expression of lower degree into that of the higher degree. If the first terms of the two expressions are of the same degree, as is the case in the example worked here, we take for divisor the one whose first term has the smaller coefficient. When the remainder from the first division is obtained, this is used as the new divisor, and the old divisor becomes the new dividend. This process is continued until a divisor divides exactly into its dividend. Such divisor is the H.C.F. of the two original expressions.

The process shown gives the required H.C.F. For any line in the working is the result of subtracting multiples of the two original expressions. Hence it follows from (2) § 133 that *every* common factor of the original expressions is also a common factor of the divisor and dividend at any stage. Hence the H.C.F. of the original expressions is also the H.C.F. of every divisor and dividend. **Thus the divisor which divides exactly into its dividend is the required H.C.F.**

The above work is usually set out as follows :

$$\begin{array}{r|r|r|l}
2a & 6a^3-15a^2+a+5 & 12a^3-27a^2-7a+15 & 2 \\
 & 6a^3-18a^2+10a & 12a^3-30a^2+2a+10 & \\
\hline
1 & 3a^2-9a+5 & 3a^2-9a+5 & \\
 & 3a^2-9a+5 & &
\end{array}$$

In this arrangement the labour involved in rewriting a previous divisor when it is to be used as the new dividend is avoided by alternating the direction in which the division is being made.

Note.—In carrying out the above process, **it is essential that both expressions should first be written in descending (or ascending) order.**

Example 7. Find the H.C.F. of $10x^3-27x^2+17x-3$ and $30x^3-61x^2+7x+3$.

$$
\begin{array}{r|r|r|l}
2x & 10x^3-27x^2+17x-3 & 30x^3-61x^2+7x+3 & 3 \\
 & 10x^3-22x^2+6x & 30x^3-81x^2+51x-9 & \\
\hline
-1 & -5x^2+11x-3 & 4)\underline{20x^2-44x+12} & \quad .\ \ (A) \\
 & -5x^2+11x-3 & 5x^2-11x+3 &
\end{array}
$$

The required H.C.F. is $5x^2-11x+3$.

Note.—In line (A) it is seen that the expression contains the numerical factor 4. Since 4 is not a factor of both the original expressions it can be removed.

When the expressions contain simple factors, these should be removed first, and their H.C.F. found. The H.C.F. of the remaining portions of the expressions should then be obtained.

Example 8. Find the H.C.F. of $12a^5b+30a^4b^2+3a^3b^3-18a^2b^4$ and $56a^6b^3+168a^5b^4+28a^4b^5-105a^3b^6$.

Removing the simple factors, the expression becomes:

$3a^2b(4a^3+10a^2b+ab^2-6b^3)$ and $7a^3b^3(8a^3+24a^2b+4ab^2-15b^3)$.

The H.C.F. of $3a^2b$ and $7a^3b^3$ is a^2b.

We next obtain the H.C.F. of the remaining portions, thus:

$$
\begin{array}{r|r|r|l}
a & 4a^3+10a^2b+ab^2-6b^3 & 8a^3+24a^2b+4ab^2-15b^3 & 2 \\
 & 4a^3+2a^2b-3ab^2 & 8a^3+20a^2b+2ab^2-12b^3 & \\
\hline
2b & 8a^2b+4ab^2-6b^3 & b)\underline{4a^2b+2ab^2-3b^3} & \quad (A) \\
 & 8a^2b+4ab^2-6b^3 & 4a^2+2ab-3b^2 &
\end{array}
$$

The H.C.F. of the remaining portions is $4a^2+2ab-3b^2$.

Hence the H.C.F. of the original expressions is

$$a^2b(4a^2+2ab-3b^2), \text{ that is } 4a^4b+2a^3b^2-3a^2b^3.$$

Note.—In line (A), the factor b has been removed; just as in the previous example the numerical factor 4 was removed. This

is permissible, since it is seen that b is not a factor of both the remaining portions whose H.C.F. is being found.

The questions in Exercise 18d, page 275, should now be attempted.

135. Example 9. Find the H.C.F. of

$$75x^2-45x+60x^4+225x^3 \text{ and } 145x^4-30x^2+35x^3+60x^5.$$

Rewriting the expressions in descending order:

$$60x^4+225x^3+75x^2-45x, \quad 60x^5+145x^4+35x^3-30x^2.$$

Removing the simple factors:

$$15x(4x^3+15x^2+5x-3), \quad 5x^2(12x^3+29x^2+7x-6).$$

The H.C.F. of the simple factors $15x$ and $5x^2$ is $5x$.

We now proceed to find the H.C.F. of the remaining portions.

	$4x^3+15x^2+5x-3$	$12x^3+29x^2+7x-6$	3
(A)	4	$12x^3+45x^2+15x-9$	
$-x$	$16x^3+60x^2+20x-12$	$-16x^2-8x+3$	$-4x$
	$16x^3+8x^2-3x$	$-16x^2-12x$	
	$52x^2+23x-12$	$4x+3$	1
(B)	4	$4x+3$	
-13	$208x^2+92x-48$		
	$208x^2+104x-39$		
(C)	$-3)\ -12x-9$		
	$4x+3$		

The H.C.F. of the remaining portions is $(4x+3)$. Thus the H.C.F. of the original expressions is $5x(4x+3)$, that is $20x^2+15x$.

Note.—When the first remainder has been obtained, we should divide $-16x^2$, etc., into $4x^3$, etc. This would give the fractional quotient of $-\frac{1}{4}x$. To avoid this fraction, the dividend has been multiplied in line (A) by 4. If as a consequence of this we find that the H.C.F. contains the numerical factor 4, this factor would be rejected, since we have already shown that the numerical factor of the required H.C.F. is 5. Similarly, in line (B) the dividend has been multiplied by 4 to avoid the fractional quotient that would be obtained on dividing $-16x^2$, etc., into $52x^2$, etc. In line (C) the numerical factor -3 has been removed.

136. Example 10. Find the H.C.F. of $6x^3-x^2-32x+20$,

$$12x^3+4x^2-53x+30,$$

and $2x^3+11x^2+x-35$.

The H.C.F. of the first two expressions contains all factors common to the two. Thus if the H.C.F. of this result and the third expression be found, it will contain all factors common to the three expressions, and will thus be the H.C.F. of the three.

$$\begin{array}{r|r|r|l}
x & 6x^3-x^2-32x+20 & 12x^3+4x^2-53x+30 & 2 \\
 & 6x^3+11x^2-10x & 12x^3-2x^2-64x+40 & \\
\cline{2-3}
-2 & -12x^2-22x+20 & 6x^2+11x-10 & \\
 & -12x^2-22x+20 & & \\
\end{array}$$

The H.C.F. of the first two expressions is $6x^2+11x-10$.

We now have to find the H.C.F. of

$$6x^2+11x-10 \text{ and } 2x^3+11x^2+x-35.$$

Now $6x^2+11x-10=(2x+5)(3x-2)$.

It will be found that $2x+5$ is a factor of $2x^3+11x^2+x-35$, but $3x-2$ is not.

Thus the H.C.F. of the three expressions is $2x+5$.

Example 11. Reduce to its lowest terms the fraction

$$\frac{4x^5+4x^4-15x^3-21x+4}{6x^4-19x^3+18x^2-17x+20}.$$

The H.C.F. of the numerator and denominator contains all their common factors. Hence if the numerator and denominator be divided by their H.C.F., the fraction will be reduced to its lowest terms.

$$\begin{array}{r|r|r|l}
3x & 6x^4-19x^3+18x^2-17x+20 & 4x^5+4x^4-15x^3-21x+4 & \\
 & 6x^4-9x^3+3x^2-12x & 3 & \\
\cline{2-3}
-5 & -10x^3+15x^2-5x+20 & 12x^5+12x^4-45x^3-63x+12 & 2x \\
 & -10x^3+15x^2-5x+20 & 12x^5-38x^4+36x^3-34x^2+40x & \\
\cline{2-3}
 & & 50x^4-81x^3+34x^2-103x+12 & \\
 & & 3 & \\
\cline{3-3}
 & & 150x^4-243x^3+102x^2-309x+36 & 25 \\
 & & 150x^4-475x^3+450x^2-425x+500 & \\
\cline{3-3}
 & & 116)\underline{232x^3-348x^2+116x-464} & \\
 & & 2x^3-3x^2+x-4 & \\
\end{array}$$

Thus the H.C.F. of numerator and denominator is $2x^3-3x^2+x-4$. On dividing numerator and denominator by this H.C.F., it will be found that the fraction reduces to $\dfrac{2x^2+5x-1}{3x-5}$.

Example 12. Find the H.C.F. of $8x^4+4x^3-38x^2+11x+15$ and $12x^4+28x^3-27x^2-43x+30$, and thus factorise each expression completely.

	$8x^4+4x^3-38x^2+11x+15$	$12x^4+28x^3-27x^2-43x+30$	
	11	2	
$2x$	$88x^4+44x^3-418x^2+121x+165$	$24x^4+56x^3-54x^2-86x+60$	3
	$88x^4+120x^3-238x^2+30x$	$24x^4+12x^3-114x^2+33x+45$	
	$-76x^3-180x^2+91x+165$	$44x^3+60x^2-119x+15$	$22x$
	11	$44x^3+66x^2-110x$	
-19	$-836x^3-1980x^2+1001x+1815$	$-6x^2-9x+15$	-3
	$-836x^3-1140x^2+2261x-285$	$-6x^2-9x+15$	
	$-420)\ -840x^2-1260x+2100$		
	$2x^2+3x-5$		

The H.C.F. of the two expressions is $2x^2+3x-5$.

Dividing each expression by this H.C.F., the quotients are found to be $4x^2-4x-3$ and $6x^2+5x-6$ respectively.

Thus the expressions are $(2x^2+3x-5)(4x^2-4x-3)$,
and $(2x^2+3x-5)(6x^2+5x-6)$.

Factorising the brackets, the expressions become

$$(x-1)(2x+5)(2x+1)(2x-3) \text{ and } (x-1)(2x+5)(2x+3)(3x-2).$$

The questions in Exercise 18e, page 276, should now be attempted.

Exercise 18a

Find the H.C.F. of:

1. $xy,\ xy^3$.
2. $x^2y,\ x^3y^2$.
3. $2a^2b^2,\ 6a^3b^4$.
4. $15x^2y^5,\ 27xy^3$.
5. $4pq^3,\ 7p^2q^2$.
6. $6a^2bc,\ 5ab^2$.
7. $9x^2yz^2,\ 12xz^3$.
8. $12ab^3c^4,\ 20b^2c^3$.
9. $a^2b^3c^2,\ ab^2c^3$.
10. $x^2y^3,\ xy^5,\ x^4y^2$.
11. $4a^3b^3,\ 10a^2b^5,\ 6a^4b$.
12. $6c^5d^2,\ 18cd^3,\ 15c^2d^2$.
13. $6x^2yz^2,\ 14xy^2z^3,\ 7x^3yz$.
14. $9ab^3c^2,\ 15a^2b^4c^3,\ 10a^2c^2$.
15. $8p^4q^5r^6,\ 12p^3q^6r^4,\ 20p^2q^3r^5$.
16. $10x^3yz^2,\ 35x^2y^6z^4,\ 15x^4y^3z^3$.
17. $7p^2q^3r^4,\ 10p^3q^2r^5,\ 12pq^3$.
18. $8a^5b^2c^3,\ 4a^3b^6c^4,\ 14b^3c^6$.
19. $9a^2b^3cd^4,\ 12a^3bc^3d^2,\ 5a^4b^4cd^3,\ 8a^2bc^2d^2$.
20. $4p^2q^3r^4s^2,\ 10p^3q^2s^3,\ 9p^2q^4s^3,\ 7q^5s^4$.

Exercise 18b

Find the H.C.F. of:

1. $x(x+y)$, $2y(x+y)$.
2. $2ab(a+b)$, $3b^2(a+b)$.
3. x^2-y^2, $x(x+y)$.
4. $4xy(x^2-y^2)$, $6y^2(x-y)$.
5. $6c^2d^3(c+d)$, $9cd^2(c^2-d^2)$.
6. $a^2(1+2a)$, $2a(1-4a^2)$.
7. $3xy(x^2-4y^2)$, $2y^2(x-2y)$.
8. $2cd^2(2c+3d)$, $3c^2d(4c^2-9d^2)$.
9. $3ab^2(a-b)^2$, $a^4-a^2b^2$.
10. $6x^3+9x^2y$, $4xy^2+6y^3$.
11. $8a^3+12a^2b$, $8a^2b-18b^3$.
12. $8x^2y^2(x-y)^2$, $12xy^3(x-y)$.
13. x^3+1, x^2+4x+3.
14. $8a^3-27$, $6a^2-5a-6$.
15. $(x-2)^3$, x^3-8.
16. $27c^3-1$, $3c(9c^2-1)$.
17. $6a^2b(a+b)^2$, $8a^2b^3(a+b)^3$.
18. $2ab(a-3b)^2$, $3b^2(a^2-ab-6b^2)$.
19. $x(x+y)(x-y)^2$, $y(x-y)(x+y)^2$.
20. $8d^3-1$, $4d^3+2d^2+d$.
21. $4x^2(x+4)^2$, $10x(2x^2+3x-20)$.
22. $a^3b^2+3a^2b^3$, $a^5(a^2-9b^2)$.
23. c^4-8cd^3, $2d(c-2d)^2$.
24. $6pq+18q^2$, $4p^3-36pq^2$.
25. x^2+6x+9, x^2+5x+6.
26. a^2+a-20, a^2-a-12.
27. $m^2+4m-21$, $m^2-3m-70$..
28. $c^2+3c-18$, $c^2-7c+12$.
29. $p^2+5p-24$, p^2+p-56.
30. x^3+8x^2+15x, $x^4+3x^3-10x^2$.
31. $a^4+5a^3-36a^2$, a^3+2a^2-24a.
32. $x^2+7xy+12y^2$, $x^2-2xy-15y^2$.
33. $a^2+4ab-5b^2$, $a^2+8ab+15b^2$.
34. $p^2+10pq+24q^2$, $p^2+pq-30q^2$.
35. $x^3-2x^2y-15xy^2$, $x^2y+9xy^2+18y^3$.
36. $6a^2+17a+12$, $8a^2+10a-3$.
37. $12m^2-5m-3$, $20m^2-7m-6$.
38. $15x^2+14x-8$, $20x^2-43x+14$.
39. $28b^2+39b-54$, $20b^2+29b-36$.
40. $12c^3-23c^2d-24cd^2$, $15c^4-34c^3d-16c^2d^2$.
41. x^2y^3-y, $3x^2y^2-xy-2$.
42. $6a^2+11ab-10b^2$, $(3ab-2b^2)^2$.
43. a^3-b^3, $(a-b)^2$, a^2-ab.
44. $2x+y$, $4x^2-y^2$, $8x^3+y^3$.
45. x^2-16, x^2-x-12, $2x^2-3x-20$.
46. $6x^2-4x$, $9x^2+6x-8$, $12x^2+7x-10$.
47. $12x^3+3x^2$, $8x^3-10x^2-3x$, $16x^3-x$.
48. $8a^2-6ab-35b^2$, $6a^2+ab-40b^2$, $10a^2-ab-60b^2$.

Exercise 18c

Find the H.C.F. of:

1. $x^2+3x-10$, x^3+2x^2-x-14.
2. $3a^2-13a+12$, $a^3+2a^2-24a+27$.
3. $6y^2+5y-21$, $6y^3-25y^2+36y-18$.

4. x^3-64, $x^3+3x^2-31x+12$.
5. $8a^3-27$, $6a^3+a^2-21a+9$.
6. y^4+8y, $3y^3+10y^2+3y-10$.
7. $27p^4-p$, $12p^4+14p^3-15p^2+3p$.
8. $8x^4+xy^3$, $8x^4+10x^3y-x^2y^2-2xy^3$.
9. m^4-64m, $m^3+m^2+4m-48$.
10. $a^2+3a-10$, $2a^2-a-6$, $4a^3-5a^2-8a+4$.
11. $2b^2-5b-12$, $4b^2-4b-15$, $6b^3+17b^2+4b-12$.
12. $12y^2-11y+2$, $4y^2+3y-1$, $8y^3+18y^2-29y+6$.
13. $3x^2+xy-2y^2$, $6x^2-xy-2y^2$, $6x^3-x^2y-11xy^2+6y^3$.
14. $4a^2+4a-15$, $6a^2-a-12$, $4a^4-19a^2+19a-6$.
15. $6d^2+13d+6$, $6d^2-11d-10$, $12d^4-d^3+9d^2+16d+4$.
16. $2x^2-x-15$, $2x^3+19x^2+29x-15$, $4x^4+16x^3-x^2-36x+10$.

Exercise 18d

Find the H.C.F. of:

1. $x^3-8x^2+17x-6$, x^3-3x^2-8x+4.
2. $2a^3-a^2-31a+35$, $2a^3+9a^2-a-15$.
3. $3y^3-17y^2+38y-30$, $3y^3-10y^2+10y+12$.
4. m^3-7m-6, $m^3-13m-12$.
5. $2b^3-7b^2+7b-12$, $4b^3-12b^2+13b-20$.
6. $p^3-5p^2-8p+48$, $p^3-5p^2-2p+24$.
7. $4x^3-4x^2-27x+30$, $2x^3-5x^2-18x+24$.
8. $3a^3+5a^2-7a+15$, $12a^3-25a^2+32a-15$.
9. $3m^3-4m^2-8m+8$, $6m^3-5m^2-14m+12$.
10. $2y^3-y^2-17y+12$, $4y^3-4y^2-27y+20$.
11. $8x^3-18x^2+19x+6$, $4x^3-4x^2-3x+18$.
12. $8p^3-6p^2-23p+21$, $4p^3-17p^2-22p+35$.
13. $3a^4-4a^3-8a$, $9a^4+3a^3+10a^2-4a$.
14. $5x^3-13x^2-10x+12$, $10x^4-x^3-10x^2+4x$.
15. $y^5+y^4-16y^3-16y^2$, $5y^4-14y^3-23y^2-4y$.
16. $m^3-13m+12$, $m^3+2m^2-9m-18$.
17. $4x^4+6x^3-26x^2-24x$, $16x^4-20x^3-38x^2-8x$.
18. $7x^3+26x^2+6x-4$, $4x^5+4x^4-40x^3-24x^2$.
19. $p^3+10p^2+31p+30$, $3p^4+24p^3+33p^2-60p$.
20. $a^3-2a^2b-18ab^2-9b^3$, $8a^3-39a^2b-29ab^2-3b^3$.

21. $4x^3-4x^2y-17xy^2-7y^3,\ 2x^3+3x^2y-16xy^2-21y^3.$
22. $2p^3-p^2q-11pq^2-12q^3,\ 6p^3+11p^2q+2pq^2-8q^3.$
23. $2x^4-2x^3y-36x^2y^2+48xy^3,\ 18x^4+50x^3y-120x^2y^2+24xy^3.$
24. $x^4y-x^3y^2-22x^2y^3-8xy^4,\ 10x^3y-46x^2y^2-40xy^3-8y^4.$

Exercise 18e

Find the H.C.F. of:

1. $3x^5+6x^4-27x^3-54x^2,\ 6x^4+12x^3-90x^2-216x.$
2. $8a^4+56a^3+128a^2+96a,\ 12a^5+60a^4+24a^3-96a^2.$
3. $y^3+8y^2+4y-48,\ y^3+10y^2+17y-28.$
4. $4b^5-48b^3+64b^2,\ 2b^4+8b^3-14b^2-20b.$
5. $4x^3-16x^2-9x+36,\ 6x^3-23x^2+9x+18.$
6. $a^3+5a^2b-13ab^2+7b^3,\ a^3+3a^2b-16ab^2+12b^3.$
7. $x^3+8x^2y+4xy^2-48y^3,\ x^3+12x^2y+41xy^2+30y^3.$
8. $12x^5y+23x^4y^2-29x^3y^3-60x^2y^4,\ 8x^4y^2-6x^3y^3-89x^2y^4-60xy^5.$
9. $a^3-a^2-32a+60,\ a^4+3a^3-35a^2-39a+70.$
10. $b^4+5b^3-13b^2-77b-60,\ b^3-2b^2-16b+32.$
11. $6m^4-m^3-83m^2-54m+72,\ 9m^4+9m^3-112m^2-4m+48.$
12. $24x^4-22x^3-87x^2+43x+84,\ 32x^4-144x^3+6x^2+211x+84.$
13. $40a^4-274a^3b+445a^2b^2-87ab^3-54b^4,$
 $48a^4-256a^3b+385a^2b^2-127ab^3-60b^4.$
14. $63y^5+267y^4+223y^3-67y^2-110y-24,$
 $54y^4-57y^3-163y^2+98y+48.$
15. $41b^2-296b^3-189+60b^4+384b,\ 43b-199b^2+140-30b^3+72b^4.$
16. $593x^3+96-208x+168x^4-374x^2,$
 $144x^4-153x+178x^3+72-191x^2.$
17. $6a^3+11a^2-19a+6,\ 12a^3+25a^2-31a+6,\ 8a^3+18a^2-17a+3.$
18. $24m^4-22m^3-57m^2+45m,\ 12m^3+4m^2-31m-15,$
 $24m^4-46m^3+m^2+15m.$
19. $8x^5-30x^4+31x^3-6x^2,\ 4x^5-21x^4+29x^3-6x^2,$
 $8x^4-38x^3+45x^2-9x.$
20. $48a^3-116a^2b-172ab^2-35b^3,\ 36a^3-72a^2b-169ab^2-70b^3,$
 $72a^3+126a^2b+67ab^2+10b^3.$
21. $30y^3-86y+40-47y^2,\ 101y^2+10y+60y^3-24,$
 $30-47y+40y^3-86y^2.$

22. $6x^4-5x^3-75x^2-10x+24$, $12x^4+56x^3+59x^2-9x-18$,
$12x^4-28x^3-81x^2-2x+24$, $6x^4+7x^3-79x^2-162x-72$.

23. Find the H.C.F. of $48x^4-76x^3-320x^2-37x+105$
and $72x^4-126x^3-437x^2-24x+35$,
and then proceed to factorise each expression completely.

24. Reduce to its lowest terms the fraction

$$\frac{4x^4-16x^3+27x^2-22x+8}{6x^4-5x^3-10x^2+19x-10}.$$

25. Reduce to its lowest terms the fraction

$$\frac{12a^4-43a^3b+61a^2b^2-39ab^3+10b^4}{27a^4-42a^3b-16a^2b^2+63ab^3-20b^4}.$$

CHAPTER XIX

Lowest Common Multiple

137. The lowest common multiple (L.C.M.) of two or more algebraical expressions, is the expression of lowest degree which is exactly divisible by each of them.

Example 1. Find the L.C.M. of x^3, x^8, x^2, x^5.

The only letter which occurs is x. The expression of lowest dimensions into which each divides exactly is x^8.

Hence the L.C.M. is x^8.

Example 2. Find the L.C.M. of x^2y^6, x^5y^3, xy^4.

The lowest power of x that is divisible by x^2, x^5 and x, is x^5.

The lowest power of y that is divisible by y^6, y^3 and y^4, is y^6.

Hence the expression of lowest dimensions divisible by the three given expressions, is x^5y^6, which is thus the L.C.M.

Example 3. Find the L.C.M. of $54a^3b^4x^2$, $30a^6b^3x^6$, $45a^5bx^7$.

The L.C.M. of 54, 30, 45 is 270.

The lowest power of a that is divisible by a^3, a^6, a^5, is a^6.

The lowest power of b that is divisible by b^4, b^3, b, is b^4.

The lowest power of x that is divisible by x^2, x^6, x^7, is x^7.

Hence the expression of lowest dimensions divisible by the three given expressions is $270a^6b^4x^7$, which is thus the L.C.M.

The questions in Exercise 19a, page 281, should now be attempted.

138. If the expressions can be readily factorised, their L.C.M. can be found as follows:

Example 4. Find the L.C.M. of $8b(a^2-b^2)$, $6ab^3(a^3-b^3)$, $18a^2(a+b)^2$.

The L.C.M. of 8, 6, 18, is 72.

The L.C.M. of the simple factors b, ab^3, a^2, is a^2b^3.

Factorising the remaining portions,

$$a^2-b^2=(a+b)(a-b).$$
$$a^3-b^3=(a-b)(a^2+ab+b^2).$$
$$(a+b)^2=(a+b)^2.$$

The L.C.M. of these remaining portions is $(a+b)^2(a-b)(a^2+ab+b^2)$.

Thus the L.C.M. of the original expressions is

$$72a^2b^3(a+b)^2(a-b)(a^2+ab+b^2).$$

The pupil should verify that the L.C.M., if required free of brackets, is $72a^7b^3+144a^6b^4+72a^5b^5-72a^4b^6-144a^3b^7-72a^2b^8$.

Example 5. Find the L.C.M. of
$14x^4-14x^2y^2$, $18x^3y-30x^2y^2+12xy^3$, $21x^3y^2-21y^5$, $2x^3-4x^2y+2xy^2$.

We first factorise the expressions:

$$14x^4-14x^2y^2=14x^2(x^2-y^2)=14x^2(x+y)(x-y).$$

$$18x^3y-30x^2y^2+12xy^3=6xy(3x^2-5xy+2y^2)=6xy(x-y)(3x-2y).$$

$$21x^3y^2-21y^5=21y^2(x^3-y^3)=21y^2(x-y)(x^2+xy+y^2).$$

$$2x^3-4x^2y+2xy^2=2x(x^2-2xy+y^2)=2x(x-y)^2.$$

The L.C.M. of 14, 6, 21, 2, is 42.

The L.C.M. of x^2, xy, y^2, x, is x^2y^2.

The L.C.M. of the remaining portions is

$$(x+y)(x-y)^2(3x-2y)(x^2+xy+y^2).$$

Thus the L.C.M. of the given expressions is

$$42x^2y^2(x+y)(x-y)^2(3x-2y)(x^2+xy+y^2).$$

Note.—It is usually convenient to leave a result such as this L.C.M. in the factorised form.

The questions in Exercise 19b, page 281, should now be attempted.

General Method for L.C.M.

139. When it is required to find the L.C.M. of two expressions which cannot be readily factorised, the following principle may be applied.

Let A and B denote the two expressions, and h their H.C.F.

Denote the quotients $\frac{A}{h}$, $\frac{B}{h}$, by Q and R respectively.

Then $A=hQ$, and $B=hR$.

As h is the highest common factor of A and B, that is of hQ and hR, it follows that Q and R can have no common factor.

Hence the expression of lowest dimensions into which hQ and hR divide is hQR.

Thus the L.C.M. of A and B is $hQR=\frac{hQ\times hR}{h}=\frac{A\times B}{h}$. (1)

Thus **the L.C.M. of any two expressions may be found by dividing their product by their H.C.F.** Now the H.C.F. of any two expressions can always be found by the methods explained in Chapter XVIII; hence by use of this principle their L.C.M. can also be obtained.

Note.—From line (1) we see that h times the L.C.M. of A and $B=A\times B$. Thus **the product of two expressions is equal to the product of their H.C.F. and their L.C.M.**

Example 6. Find the L.C.M. of

$$2x^4+9x^3+6x^2+6x-35 \text{ and } 2x^4+11x^3+x^2-37x+20.$$

It will be found that the H.C.F. of the two expressions is x^2+3x-5.

By the above principle, the L.C.M. of the two expressions is

$$\frac{(2x^4+9x^3+6x^2+6x-35)(2x^4+11x^3+x^2-37x+20)}{x^2+3x-5}.$$

It will be found on division that

$$\frac{2x^4+9x^3+6x^2+6x-35}{x^2+3x-5}=2x^2+3x+7.$$

Hence the L.C.M. of the two expressions

$$=(2x^2+3x+7)(2x^4+11x^3+x^2-37x+20).$$

As x^2+3x-5 is the H.C.F. of the two original expressions, it must divide into $2x^4+11x^3+x^2-37x+20$.

It will be found that

$$2x^4+11x^3+x^2-37x+20=(x^2+3x-5)(2x^2+5x-4).$$

Thus the L.C.M. can be written in the alternative form

$$(2x^2+3x+7)(x^2+3x-5)(2x^2+5x-4).$$

140. If it is required to find the L.C.M. of more than two expressions, A, B and C say, then the L.C.M. of A and B can first be found by the method described in § 139. This will be the expression of lowest dimensions into which A and B will each divide. Hence if the L.C.M. of this result and of C be now found, it will be the expression of lowest dimension into which all three will divide, that is the L.C.M. of A, B and C.

The questions in Exercise 19c, page 282, should now be attempted.

Exercise 19a

Find the L.C.M. of:

1. x^3, xy. **2.** $x^2, 2xy$. **3.** $a^3, 2abc$.
4. $2a^2b, 3abc$. **5.** $2ab^2, 5a^2bc$. **6.** $6xy^2z, 4xyz^2$.
7. $7ac^3, 3a^2b$. **8.** $8x^3yz^4, 12x^2y^2z^2$. **9.** $9p^2q^3r^4, 6p^3q^2r$.
10. $2a^2x, 3aby$. **11.** $4ab^2y, 10bx^2y^3$. **12.** $3x, 2y, 4z$.
13. $6a^2, 4b^2, 12ab$. **14.** ab, bc, ac. **15.** $2xy, 3xz^2, 6yz$.
16. $5ab^2, 8bc^3, 10ac^2$. **17.** a^2bc, b^2ca, c^2ab.
18. $6xy^2z, 7x^2yz^3, 9x^3z^3$. **19.** $10p^3q^4r, 15pq^2r^3, 20p^2q^3r^4$.
20. $8a^2bx, 12bcy^2, 18ac^2xy$. **21.** $9c^2xy^2, 15cdy^3z, 6d^3x^2z^2$.
22. $12a^2pq^3, 15b^2cp^2q, 10ac^2pq^2$. **23.** $6ab^3, 4bd^2, 8a^2c^3, 9ab^2d$.
24. $7w^2xz^3, 12wy^3z, 8x^4yz^4, 42w^2y^2z^3$.

Exercise 19b

Find the L.C.M. of:

1. x^2, x^2+x. **2.** x^3, x^2-3x. **3.** $4y^2, 6y^2-6y$.
4. $8m^3, 10m^2+20m$. **5.** $2a+2b, a^2+ab$. **6.** $(x+y)^2, x^2y+xy^2$.
7. $2a^2b, 4a^2+a$. **8.** x^2+2x, x^2-4. **9.** p^3-p, p^2+p.
10. a^2-b^2, a^3-b^3. **11.** $(a-2)^2, a^2-4$. **12.** $4x^2y, 3x^2y+4xy^2$.
13. $(a+b)^2, a^3+b^3$. **14.** $(m-3)^2, m^3-27$. **15.** $y^3-4y^2, (y-4)^3$.
16. $y^4-4y^2, (y+2)^2$. **17.** x^2-4, x^2+x-6.
18. x^2-3x-4, x^2+2x+1. **19.** a^2-2a-8, a^2+a-2.
20. y^3-1, y^2+4y-5. **21.** a^2-9b^2, a^2+3ab.
22. $x^2+2x-15, x^2+x-12$. **23.** $p^2+pq-20q^2, p^2-3pq-4q^2$.
24. $x^2+5x+6, x^2-3x-10, x^2-2x-15$.
25. $a^2+10a+24, a^2+3a-4, a^2+5a-6$.
26. $y^2-2y-3, y^2+6y+5, y^2+2y-15$.
27. $p^2-2pq-3q^2, p^2+7pq+6q^2, p^2+3pq-18q^2$.
28. $2x^2+5x-12, 4x^2-16x+15, 2x^2+3x-20$.

29. $3m^2+11m+6,\ 6m^2-17m-14,\ 2m^2-m-21.$
30. $x-y,\ x^3+y^3,\ x^3-y^3,\ x^2-xy+y^2,\ x^2-y^2.$
31. $8a^2-2a-3,\ 12a^2-17a+6,\ 6a^2-a-2.$
32. $4x^2+17xy+15y^2,\ 12x^2-xy-20y^2,\ 3x^2+5xy-12y^2.$
33. $12a^3+17a^2-5a,\ 20a^2-13a+2,\ 15a^3+19a^2-10a.$
34. $6x^2y(x-y)^3,\ 8xy(x+y)^3,\ 9xy^2(x^2-y^2).$
35. $(3a^2-2ab)^2,\ 3a^2-8ab+4b^2,\ (a^3-4ab^2)^2.$
36. $8cd^3(2c^2-cd),\ 12c^2d(8c^3-d^3),\ 6d^2(4c^2d-d^3).$

Exercise 19c

Find the L.C.M. of:

1. $x^3+3x^2+4x+2,\ x^3-x^2-4x-6.$
2. $3x^3-10x^2+10x-4,\ 6x^3-11x^2+8x-2.$
3. $8a^3-2a^2+3a+3,\ 12a^3-17a^2+15a-6.$
4. $6p^3-29p^2q+45pq^2-25q^3,\ 3p^3+2p^2q-16pq^2+15q^3.$
5. $18y^4+9y^3+y^2-2y,\ 12y^5+52y^4+39y^3+14y^2.$
6. $32a^3-14ab^2-3b^3,\ 24a^3-62a^2b+5ab^2+21b^3.$
7. $15x^3-61x^2y-78xy^2+40y^3,\ 30x^3+13x^2y-30xy^2+8y^3.$
8. $28c^4+44c^3d-89c^2d^2-60cd^3,\ 56c^3d-122c^2d^2+17cd^3+60d^4.$
9. $x^3+7x^2+16x+12,\ 3x^3+15x^2+6x-24.$
10. $y^4-12y^2+16y,\ y^4+4y^3-7y^2-10y.$
11. $12x^3+25x^2-31x+6,\ 8x^3+18x^2-17x+3,\ 6x^3+11x^2-19x+6.$
12. $12a^3+4a^2-31a-15,\ 24a^3-46a^2+a+15,\ 24a^3-22a^2-57a+45,$ $16a^3+20a^2-12a-9.$
13. $4x^3-21x^2y+29xy^2-6y^3,\ 8x^3-30x^2y+31xy^2-6y^3,$ $8x^3-38x^2y+45xy^2-9y^3.$
14. $24m^3-62m^2-73m-14,\ 72m^3+126m^2+67m+10,$ $48m^3-116m^2-172m-35,\ 36m^3-72m^2-169m-70.$
15. $24x^3-10x^2-101x-60,\ 60x^4+101x^3+10x^2-24x,$ $40x^3-86x^2-47x+30,\ 30x^5-47x^4-86x^3+40x^2.$
16. Find two expressions of the second degree whose H.C.F. is $a+2b$ and whose L.C.M. is $6a^3+5a^2b-12ab^2+4b^3$.
17. Find two expressions of the third degree whose H.C.F. is x^2+y^2 and whose L.C.M. is x^4-y^4.
18. Find two expressions of the fourth degree whose H.C.F. is $3x^2-2x+1$ and whose L.C.M. is

$$18x^6-12x^5-15x^4+14x^3-16x^2+6x-3.$$

CHAPTER XX

Square Root

141. The **square root** of a given expression is that quantity which when raised to the second power, that is when squared, becomes the given expression.

Thus, the square root of x^8 is x^4; because $(x^4)^2=x^8$.

The **nth root** of a given expression is that quantity which when raised to the nth power becomes the given expression.

Thus, the fifth root of a^{15} is a^3; because $(a^3)^5=a^{15}$.

The symbol employed to indicate an nth root is $\sqrt[n]{\ }$.

The symbol for a square root is written $\sqrt{\ }$, without the figure 2.

The symbol $\sqrt[3]{\ }$ denotes a cube root.

The symbol $\sqrt{\ }$ is called the **radical sign,** and the number is called the **order.**

Since $(-x)^2=(-x)(-x)=x^2$, we see that the square of $-x$ is the same as the square of $+x$. It follows that the square root of x^2 can be $+x$ or $-x$.

Thus, $\sqrt{36}$ is $+6$ or -6.

Similarly, **if the order of a root is any even number, the root has two values equal in magnitude but opposite in sign.**

Thus, $\sqrt[6]{729}$ is $+3$ or -3.

Example 1. $\sqrt{25a^4b^{10}}=+5a^2b^5$ or $-5a^2b^5$; because both $(+5a^2b^5)^2$ and $(-5a^2b^5)^2=25a^4b^{10}$.

The two roots may be conveniently written $\pm 5a^2b^5$; the sign $\pm$ is read 'plus or minus.'

Any odd power of a quantity has the same sign as the quantity itself.

e.g. $(+7)^3=(+7)(+7)(+7)=343$; $(-7)^3=(-7)(-7)(-7)=-343$.

Therefore if the order of a root of a quantity is any odd number, the root has the same sign as the quantity.

Example 2. $\sqrt[3]{8x^3y^6}=2xy^2$, and $\sqrt[3]{-8x^3y^6}=-2xy^2$.

The square of any quantity, positive or negative, is positive; hence only positive quantities can have square roots.

An expression such as $\sqrt{-4}$ is for our purpose meaningless, and is called an **imaginary quantity.**

In the same way, any even root of a negative quantity is imaginary.

Example 3.

$\sqrt{49a^4x^2y^6}=\pm 7a^2xy^3$;

because $(+7a^2xy^3)^2$ and $(-7a^2xy^3)^2=49a^4x^2y^6$.

$\sqrt[3]{125p^6q^3}=5p^2q$; because $(5p^2q)^3=125p^6q^3$.

$\sqrt[4]{81a^{12}b^8}=\pm 3a^3b^2$; because $(+3a^3b^2)^4$ and $(-3a^3b^2)^4=81a^{12}b^8$.

$\sqrt[5]{-32x^{10}y^{20}}=-2x^2y^4$; because $(-2x^2y^4)^5=-32x^{10}y^{20}$.

We deduce the following method for obtaining any required root of a given simple expression:

1. **Find the required root of the numerical coefficient by arithmetic, and prefix it with its proper sign.**
2. **Find the index of each literal factor by dividing the index in the given expression by the order of the root required.**

Example 4.

$$\sqrt[5]{243a^{15}b^{40}}=3a^{\frac{15}{5}}b^{\frac{40}{5}}=3a^3b^8.$$

$$\sqrt[4]{256x^{16}y^{28}}=\pm 4x^{\frac{16}{4}}y^{\frac{28}{4}}=\pm 4x^4y^7.$$

$$\sqrt[3]{\frac{-27x^6y^{21}}{343a^{12}b^9}}=-\frac{3x^{\frac{6}{3}}y^{\frac{21}{3}}}{7a^{\frac{12}{3}}b^{\frac{9}{3}}}=-\frac{3x^2y^7}{7a^4b^3}.$$

The questions in Exercise 20a, page 290, should now be attempted.

Square Root by Inspection

142. From identities (1) and (2) in § 95, we are able to write down the square of any binomial (i.e. an expression of two terms).

Thus, $(3x+4y)^2=(3x)^2+2(3x)(4y)+(4y)^2$
$=9x^2+24xy+16y^2.$

Conversely, an expression may sometimes be recognised as an exact square, and its square root written down by inspection.

Example 5. Find the square root of $49a^2+84ab+36b^2$.

$$\begin{aligned}\text{The expression}&=(7a)^2+2(42ab)+(6b)^2\\&=(7a)^2+2(7a)(6b)+(6b)^2\\&=(7a+6b)^2.\end{aligned}$$

Thus the required square root is $\pm(7a+6b)$.

For the remaining portion of this chapter, unless otherwise stated, we shall omit the $\pm$ sign when dealing with an even root, and shall confine our attention to the positive root.

Example 6. Find the square root of $25x^2y^2-30xyz+9z^2$.

$$\begin{aligned}\text{The expression}&=(5xy)^2-2(15xyz)+(3z)^2\\&=(5xy)^2-2(5xy)(3z)+(3z)^2\\&=(5xy-3z)^2.\end{aligned}$$

Thus the required square root is $5xy-3z$.

Example 7. Find the square root of $4\frac{a^4}{b^4}-12+9\frac{b^4}{a^4}$.

$$\begin{aligned}\text{The expression}&=\left(2\frac{a^2}{b^2}\right)^2-2(6)+\left(3\frac{b^2}{a^2}\right)^2\\&=\left(2\frac{a^2}{b^2}\right)^2-2\left(2\frac{a^2}{b^2}\right)\left(3\frac{b^2}{a^2}\right)+\left(3\frac{b^2}{a^2}\right)^2\\&=\left(2\frac{a^2}{b^2}-3\frac{b^2}{a^2}\right)^2.\end{aligned}$$

Thus the required square root is $2\frac{a^2}{b^2}-3\frac{b^2}{a^2}$.

Example 8. Find the square root of $x^2+2xy+y^2-6x-6y+9$.

$$\begin{aligned}\text{The expression}&=(x+y)^2-2(3x+3y)+(3)^2\\&=(x+y)^2-2(x+y)(3)+(3)^2=(x+y-3)^2.\end{aligned}$$

Thus the required square root is $x+y-3$.

The questions in Exercise 20b, page 291, should now be attempted.

General Method for Square Root

143. When the square root cannot readily be determined by inspection, the general method explained below may be applied.

We know that the square root of $a^2+2ab+b^2$ is $a+b$.

Let us consider how the *result* $a+b$ can be obtained from the *expression* $a+2ab+b^2$.

The first term of the result, a, is the square root of the first term of the expression, a^2.

Now $a^2+2ab+b^2-a^2=2ab+b^2=b(2a+b)$.

Hence the second term of the result, b, can be obtained by subtracting from the expression the square of a, and dividing the remainder by $2a+b$.

The work may be set out as follows:

$$
\begin{array}{lr|l l r}
 & & a^2+2ab+b^2(a+b & \ldots\ldots & (1) \\
 & & a^2 & \ldots\ldots & (2) \\
\hline
(A)\;\ldots & 2a+b & 2ab+b^2 & \ldots\ldots & (3) \\
 & & 2ab+b^2 & \ldots\ldots & (4) \\
\hline
\end{array}
$$

Explanation. The first term of the square root is a, that is the square root of the first term of the given expression, and is shown as the first term of the answer in line (1).

The term a is squared, line (2), and subtracted from the expression, giving the remainder in line (3).

The second term of the answer, b, in line (1), is the result of dividing the first term of the remainder in line (3) by $2a$, that is by twice the term already found.

The divisor is completed by adding this term b to $2a$, giving the divisor $2a+b$ in line (A).

The divisor in line (A) is now multiplied by the second term, b, in line (1), giving line (4).

On subtraction there is no remainder.

Thus the square root is exact, and is equal to $a+b$.

Example 9. Find the square root of $25a^2-40ab+16b^2$.

$$
\begin{array}{rl}
 & 25a^2-40ab+16b^2(5a-4b \quad . \quad . \quad . \quad . \quad (1) \\
 & 25a^2 \\
\hline
(2) \ldots \ 10a-4b & \left| \begin{array}{l} -40ab+16b^2 \\ -40ab+16b^2 \\ \hline \end{array} \right.
\end{array}
$$

Explanation. The first term of the square root is the square root of $25a^2$, that is $5a$, and is shown as the first term of the answer in line (1).

This term is now squared and subtracted from the given expression, leaving the remainder $-40ab+16b^2$ in line (2).

The first term of the divisor is now twice the term already found, that is $10a$. The quotient obtained on dividing $-40ab$ by $10a$ is $-4b$, so that $-4b$ is added to complete the divisor, as shown in line (2).

This quotient $-4b$ is also added to the answer in line (1).

On multiplying $10a-4b$ by $-4b$, and subtracting this from $-40ab+16b^2$, there is no remainder; so that the square root is exact, and is $5a-4b$.

144. The process explained in § 143 does not require a to be a single letter. Hence if two terms of a square root have already been found, the third term can be obtained by continuing the same process.

Example 10. Find the square root of $49x^4-42x^3-19x^2+12x+4$.

$$
\begin{array}{rl}
 & 49x^4-42x^3-19x^2+12x+4(7x^2-3x-2 \\
 & 49x^4 \\
\hline
14x^2-3x & \left| \begin{array}{l} -42x^3-19x^2+12x+4 \\ -42x^3+9x^2 \\ \hline \end{array} \right. \\
(1) \ldots \ 14x^2-6x-2 & \left| \begin{array}{l} -28x^2+12x+4 \\ -28x^2+12x+4 \\ \hline \end{array} \right.
\end{array}
$$

Explanation. The first two terms of the square root, namely $7x^2-3x$, are obtained as in the previous examples. Thus the divisor in line (1) commences with $2(7x^2-3x)$, that is $14x^2-6x$. The remainder at this stage is $-28x^2+12x+4$. The quotient obtained on dividing $-28x^2$ by $14x^2$ is -2. Hence the divisor in

line (1) is completed by the addition of the term -2, and -2 is also added to the answer. On multiplying the divisor by -2 and subtracting, there is no remainder; so that the square root is exact, and is $7x^2-3x-2$.

Note.—In order that the process for finding a square root should succeed, it is essential that the given expression be first arranged in either ascending or descending order.

Example 11. Find the square root of

$$24ax^3-12a^3x+36x^4-32a^2x^2+9a^4.$$

Arranging the expression in descending powers of x, and carrying out the work explained above:

$$\begin{array}{r|l} & 36x^4+24ax^3-32a^2x^2-12a^3x+9a^4(6x^2+2ax-3a^2 \\ & 36x^4 \\ \hline 12x^2+2ax & 24ax^3-32a^2x^2 \qquad . \quad . \quad . \quad . \quad . \quad (1) \\ & 24ax^3+4a^2x^2 \\ \hline 12x^2+4ax-3a^2 & -36a^2x^2-12a^3x+9a^4 \\ & -36a^2x^2-12a^3x+9a^4 \\ \hline \end{array}$$

Thus the square root is $6x^2+2ax-3a^2$.

Note.—In line (1), only the terms $24ax^3-32a^2x^2$ of the remainder have been brought down, as these are sufficient for the next step.

The questions in Exercise 20c, page 291, should now be attempted.

145. Example 12. Find the square root of

$$49x^2-\frac{56}{x}-82x+9x^4-30x^3+\frac{49}{x^2}+86.$$

In arranging the terms of the expression in descending order, each power is formed from the preceding one on division by x.

Thus x^4 is followed by x^3, x^3 by x^2, x^2 by x, x by a number, the number by $\frac{1}{x}$, and $\frac{1}{x}$ by $\frac{1}{x^2}$.

Thus it is seen that **the constant must separate the powers of x from the powers of the reciprocals of x.**

Arranging the expression in this way, and proceeding as before :

$$
\begin{array}{l|l}
 & 9x^4-30x^3+49x^2-82x+86-\frac{56}{x}+\frac{49}{x^2}\ (3x^2-5x+4-\frac{7}{x} \\
 & 9x^4 \\ \hline
6x^2-5x & -30x^3+49x^2 \\
 & -30x^3+25x^2 \\ \hline
6x^2-10x+4 & 24x^2-82x+86 \\
 & 24x^2-40x+16 \\ \hline
6x^2-10x+8-\frac{7}{x} & -42x+70-\frac{56}{x}+\frac{49}{x^2} \\
 & -42x+70-\frac{56}{x}+\frac{49}{x^2} \\ \hline
\end{array}
$$

Thus the required square root is $3x^2-5x+4-\frac{7}{x}$.

Example 13. Find a and b so that $16x^4-56x^3+ax^2-28x+b$ should be a perfect square.

Proceeding by the process for finding square root :

$$
\begin{array}{l|l}
 & 16x^4-56x^3+ax^2-28x+b(4x^2-7x+2 \\
 & 16x^4 \\ \hline
8x^2-7x & -56x^3+ax^2 \\
 & -56x^3+49x^2 \\ \hline
8x^2-14x+2 & (a-49)x^2-28x+b \qquad . \quad . \quad . \quad (1) \\
 & \qquad 16x^2-28x+4 \\ \hline
\end{array}
$$

Note.—It should be noticed that $(a-49)$ in line (1) is the coefficient of x^2, obtained on subtracting the coefficient 49 from the coefficient a.

The last term of the divisor, namely 2, is obtained by the consideration that there is to be no remainder. Now the middle term of the divisor is $-14x$, and it must therefore be multiplied by 2 in order to equal the middle term of line (1), that is $-28x$.

If the square root is to be exact, there must be no remainder. Hence $(a-49)$ must be equal to 16, and b must be equal to 4.

Thus, $a=65$ and $b=4$.

Square Root by Factors

146. When an expression can be readily factorised, its square root should be found from the factors, and not by the process of § 143.

Example 14. Find the square root of

$$(3x^2-7x-6)(6x^2-11x-10)(2x^2-11x+15).$$

Factorising each bracket, the expression becomes

$$(3x+2)(x-3)(3x+2)(2x-5)(2x-5)(x-3).$$

Rearranging these brackets, we obtain

$$(3x+2)^2(x-3)^2(2x-5)^2.$$

Hence the square root of the given expression is

$$(3x+2)(x-3)(2x-5).$$

The questions in Exercise 20d, page 292, should now be attempted.

Exercise 20a

Write down the square root of each of the following:

1. x^6.
2. a^2b^4.
3. x^4y^6.
4. $4p^4q^8$.
5. $9x^4y^{16}$.
6. $25c^{12}d^8$.
7. $16m^{10}n^{16}$.
8. $81a^4b^6c^{10}$.
9. $121x^8y^4z^{12}$.
10. $\frac{1}{100a^{10}}$.
11. $\frac{1}{64x^4y^8}$.
12. $\frac{25}{9a^4}$.
13. $\frac{16x^{16}}{9y^{36}}$.
14. $\frac{64a^4b^2}{25c^6}$.
15. $\frac{121p^6q^{14}}{81r^{10}}$.
16. $\frac{49a^{12}x^4}{144b^6y^8}$.

Write down the cube root of each of the following:

17. x^3y^9.
18. $8a^6b^6$.
19. $-27m^6n^{12}$.
20. $125c^3d^9$.
21. $-8x^9y^{15}$.
22. $216a^9b^{12}c^{18}$.
23. $729f^9g^{36}h^{81}$.
24. $-\frac{1}{64p^6q^{12}}$.
25. $\frac{64x^6y^9}{343z^{12}}$.
26. $-\frac{125a^6b^9}{27c^3d^6}$.

Write down the value of each of the following:

27. $\sqrt[3]{x^{21}y^{27}}$.
28. $\sqrt[5]{-32a^{10}b^{15}}$.
29. $\sqrt{169p^8q^{14}}$.
30. $\sqrt[6]{m^{18}n^{30}}$.
31. $\sqrt[4]{81c^8d^{20}}$.
32. $\sqrt[7]{-\frac{x^{14}}{y^{49}}}$.
33. $\sqrt[3]{\frac{1331a^{15}}{8000b^{27}}}$.
34. $\sqrt[8]{\frac{m^{16}}{256n^{40}}}$.

Exercise 20b

By inspection, find the square root of each of the following:

1. a^2+6a+9.
2. $x^2-10x+25$.
3. y^4+14y^2+49.
4. $81+18b^3+b^6$.
5. $4x^2+20x+25$.
6. $9p^4-42p^2+49$.
7. $16+24a^4+9a^8$.
8. $121-66y^5+9y^{10}$.
9. $16a^4+56a^2b^2+49b^4$.
10. $9x^2y^2-24a^2xy+16a^4$.
11. $25x^6+40abx^3+16a^2b^2$.
12. $49x^2-7xy+\frac{1}{4}y^2$.
13. $\frac{1}{16}a^4+\frac{1}{10}a^2b^4+\frac{1}{25}b^8$.
14. $9x^4+42+\frac{49}{x^4}$.
15. $121a^2x^2+\frac{16}{a^2}+88x$.
16. $36\frac{x^2}{a^2}+169\frac{y^2}{b^2}-156\frac{xy}{ab}$.
17. $(a+3)^2+6x(a+3)+9x^2$.
18. $(x-y)^4+25a^8+10a^4(x-y)^2$.
19. $x^2+4y^2+z^2-4xy+2xz-4yz$.
20. $9a^2+b^2+4c^2+6ab-12ac-4bc$.
21. $x^2+2xy+y^2-2x-2y+1$.
22. $a^2-2ab+b^2+12a-12b+36$.
23. $4x^2-8ax+4a^2-20x+20a+25$.

Exercise 20c

Find the square root of:

1. $x^4+6x^3+13x^2+12x+4$.
2. $a^4-4a^3+12a^2-16a+16$.
3. $m^4-10m^3+29m^2-20m+4$.
4. $9y^4+24y^3-20y^2-48y+36$.
5. $4b^4-28b^3+65b^2-56b+16$.
6. $25+80x+34x^2-48x^3+9x^4$.
7. $36a^4+48a^3b-116a^2b^2-88ab^3+121b^4$.
8. $16x^4-56x^3y+89x^2y^2-70xy^3+25y^4$.
9. $p^6-8p^5+20p^4-22p^3+28p^2-12p+9$.
10. $1+6m+m^2-30m^3-2m^4+24m^5+9m^6$.
11. $37x^2-6x^4-30x+x^6+9-14x^3+4x^5$.
12. $a^6-10a^5b+19a^4b^2+38a^3b^3-31a^2b^4-24ab^5+16b^6$.
13. $4x^6+12x^5y-15x^4y^2-20x^3y^3+60x^2y^4-48xy^5+16y^6$.
14. $25x^6+20ax^5-66a^2x^4-68a^3x^3+33a^4x^2+56a^5x+16a^6$.
15. $52m^3-16m+9m^6+4-30m^5-4m^2+m^4$.
16. $76c^4d^2-90cd^5+48c^5d-83c^2d^4+16c^6-12c^3d^3+81d^6$.
17. Prove that $a(a+1)(a+2)(a+3)+1$ is a perfect square, and find its square root. Use this result to find four consecutive numbers whose product is 43680.
18. Find the square root of $4x^6-12x^3y^2+13y^4+6x^3-5y^2-6x-8y$ in terms of x, where $x=y+1$.

19. Find the fourth root of

$$a^8-12a^7+58a^6-144a^5+195a^4-144a^3+58a^2-12a+1.$$

20. Find the fourth root of

$$16x^8+32x^7y-8x^6y^2-40x^5y^3+x^4y^4+20x^3y^5-2x^2y^6-4xy^7+y^8.$$

Exercise 20d

Find the square root of:

1. $x^2+6x+7-\frac{6}{x}+\frac{1}{x^2}$.

2. $4a^2-20a+37-\frac{30}{a}+\frac{9}{a^2}$.

3. $9y^2-6y+13-\frac{4}{y}+\frac{4}{y^2}$.

4. $25x^2-\frac{8}{x}+\frac{4}{x^2}-20x+24$.

5. $30m+\frac{16}{m^2}-\frac{40}{m}+9m^2+1$.

6. $16\frac{x^2}{a^2}-4\frac{a}{x}+12+\frac{a^2}{x^2}-16\frac{x}{a}$.

7. $24\frac{a}{b}+4\frac{b^2}{a^2}-16\frac{b}{a}+9\frac{a^2}{b^2}+4$.

8. $41x^2-40\frac{x^3}{a}+4a^2-20ax+16\frac{x^4}{a^2}$.

9. $4x^4+17x^2+\frac{4}{x}-2-8x+\frac{1}{x^2}-12x^3$.

10. $9a^2+13+\frac{4}{a^4}+\frac{8}{a^3}-6a+\frac{8}{a}$.

Find the values of a and b in order that the following expressions should be perfect squares:

11. $9x^4+12x^3-20x^2+ax+b$.

12. $25x^4-20x^3+ax^2-12x+b$.

13. $16x^4+24x^3+ax^2+bx+4$.

Find the values of a, b and c in order that the following expressions should be perfect squares:

14. $4x^6-4x^5+13x^4+ax^3+bx^2+6x+c$.

15. $9x^6+12x^5-8x^4+ax^3+20x^2+bx+c$.

Factorise the following expressions completely, and thus obtain their square roots:

16. $(2a^2-a-6)(2a^2-6a+4)(4a^2+2a-6)$.

17. $(6x^2-13x+6)(3x^2-11x+6)(2x^2-9x+9)$.

18. $(4m^2-4m-15)(2m^2+3m-20)(2m^2+11m+12)$.

19. $(15y^2-17y+4)(10y^2-3y-4)(6y^2+7y-3)(4y^2+8y+3)$.

20. $(2a^2-a-28)(6a^2+25a+14)(3a^2-13a-10)(a^2-9a+20)$.

CHAPTER XXI

Quadratic Equations

Solving by Factorising

147. Any equation which contains the square of an unknown quantity, but no higher power, is called a quadratic equation in that unknown, or an equation of the second degree. Thus, $x^2-8x+15=0$ is a quadratic equation in x.

Example 1. Solve the equation $x^2-8x+15=0$.

By the principles of Chapter XV, the factors of $x^2-8x+15$ are obtained from $\dfrac{1}{1}\Big|\dfrac{-3}{-5}$.

Hence the equation becomes $(x-3)(x-5)=0$.

In § 120 it was shown that the product of a number of factors cannot be zero unless one or more of them is zero.

Hence in the above example, either $(x-3)=0$, or $(x-5)=0$.

Thus, either $x=3$, or $x=5$.

There is no value of x, other than 3 or 5, which makes either factor zero, so that these are the only values of x which make the L.H.S. equal to zero. Hence they are the only values of x which satisfy the equation.

Verification. If $x=3$, then $x^2-8x+15=(3)^2-8(3)+15$
$$=9-24+15=0.$$
If $x=5$, then $x^2-8x+15=(5)^2-8(5)+15$
$$=25-40+15=0.$$

Note.—It is important to emphasise that only when the product of a number of factors is **zero** can we deduce that one or more of them is zero. If the product of two factors is other than zero, say 12, we cannot deduce that either of them is 12. For one factor may be 3 and the other 4; or one may be 2 and the other 6, etc.

It follows that **when solving an equation by the method of factors, the R.H.S. must be zero; thus all the terms of the equation must be transposed to the L.H.S. before factorising.**

148. Example 2. Solve $x^2=px.$

Before factorising, all the terms must be transposed to the L.H.S. of the equation, so that the R.H.S. is zero.

The equation becomes $x^2-px=0.$

Factorising, $x(x-p)=0.$

In order that the L.H.S. should be zero, one of the factors must be zero.

Hence either $x=0$, or $x-p=0$,

i.e. $x=0$ or p.

Note.—In the equation as originally given, the pupil might be tempted to divide the two sides of the equation by x. He would then obtain the solution $x=p$ only, thus missing the root $x=0$. It should be noticed that **if a factor containing the unknown quantity is divided into *both* sides of an equation, one root is obtained by equating that factor to zero.** Thus if $(x-3)$ is a factor of both sides of an equation, and all the terms be brought to the L.H.S., then $(x-3)$ will be a factor of the L.H.S. Hence one root of the equation is $x=3$.

A quadratic equation, such as $x^2-8x+16=0$, has two roots which are equal. For the equation may be written in the form $(x-4)^2=0$; and although 4 is the only value of x which satisfies the equation, it is convenient to state that the equation has two roots each equal to 4.

The questions in Exercise 21a, page 305, should now be attempted.

149. Example 3. Solve $(x+5)(16x^2-7x-3)=(x+5)(x^2-6x+3)$.

As $(x+5)$ is a factor of both sides of the equation, one root is obtained from $(x+5)=0$. Hence $x=-5$ is one root.

The remaining roots are obtained from $16x^2-7x-3=x^2-6x+3$.

Transposing, $15x^2-x-6=0$.

The factors are obtained from $\dfrac{3}{5}\Big|\dfrac{-2}{3}$.

$\therefore\quad (3x-2)(5x+3)=0.$

Hence either $(3x-2)=0$, or $(5x+3)=0$.

$\therefore\quad x=\frac{2}{3}$, or $-\frac{3}{5}$.

Thus the complete solution is $x=-5$, $\frac{2}{3}$, or $-\frac{3}{5}$.

The pupil should verify that each of these values satisfies the original equation.

Example 4. Solve $(3x-4)(2x+5)=(3-5x)(2x-3)$.

The first step is to multiply out all the brackets,

$$6x^2+7x-20=-10x^2+21x-9.$$

Transposing, $16x^2-14x-11=0$.

The factors are obtained from $\dfrac{2}{8}\Big|\dfrac{1}{-11}$.

$$\therefore\quad (2x+1)(8x-11)=0.$$

Hence the solution is $x=-\frac{1}{2}$, or $\frac{11}{8}$.

The student should verify that

$$\text{if } x=-\tfrac{1}{2},\ \text{L.H.S.}=-22,\ \text{R.H.S.}=-22;$$
$$\text{if } x=\tfrac{11}{8},\ \text{L.H.S.}=\tfrac{3}{3}\tfrac{1}{2},\quad \text{R.H.S.}=\tfrac{3}{3}\tfrac{1}{2}.$$

Note.—If one side of an equation has factors, but none of these factors divide exactly into the other side, these factors do not yield solutions of the equation. The factors must therefore be multiplied out as in the above example.

Example 5. Solve $\dfrac{3x+2}{x+1}-\dfrac{5x+1}{2x+1}=\dfrac{7}{12}$.

The first step is to clear of fractions by multiplying each term by the L.C.M. of the denominators, which in this example is

$$12(x+1)(2x+1).$$

We obtain $$\frac{12(x+1)(2x+1)(3x+2)}{(x+1)}-\frac{12(x+1)(2x+1)(5x+1)}{(2x+1)}$$
$$=\frac{12(x+1)(2x+1)\times 7}{12}.$$

That is, $12(2x+1)(3x+2)-12(x+1)(5x+1)=7(x+1)(2x+1)$.

Multiplying out the brackets,

$$72x^2+84x+24-60x^2-72x-12=14x^2+21x+7.$$

Transposing, $-2x^2-9x+5=0$.

Changing the sign throughout,

$$2x^2+9x-5=0.$$
$$\therefore\quad (2x-1)(x+5)=0.$$

Hence the solution is $x=\frac{1}{2}$, or -5.

Example 6. Solve $$1{\cdot}5-\frac{1{\cdot}3}{x}-\frac{2}{x^2}=0.$$

First clear of decimals by multiplying each term by 10.

$$15-\frac{13}{x}-\frac{20}{x^2}=0.$$

Multiplying each term by the L.C.M. of the denominators, which is x^2, we obtain $15x^2-13x-20=0$.

$$\therefore \quad (5x+4)(3x-5)=0.$$

Hence the solution is $x=-\frac{4}{5}$, or $\frac{5}{3}$.

The questions in Exercise 21b, page 306, should now be attempted.

Solving by Completing the Square

150. In the previous examples, the quadratic equation has been solved by factorising the L.H.S. When the quadratic expression cannot readily be factorised, a solution by the method known as 'completing the square' may be employed.

We know that $x^2+2ax+a^2=(x+a)^2$,

and $x^2-2ax+a^2=(x-a)^2$.

Now write $2a=n$,

then $$x^2+nx+\left(\frac{n}{2}\right)^2=\left(x+\frac{n}{2}\right)^2;$$

and $$x^2-nx+\left(\frac{n}{2}\right)^2=\left(x-\frac{n}{2}\right)^2.$$

We see that **if an expression consists of x^2 together with a term of the first degree in x, and we add to the expression the square of half the coefficient of x, the result is an exact square.**

Thus the term that must be added to x^2+10x to make an exact square is $(\frac{10}{2})^2$, that is 25.

For $x^2+10x+25=(x+5)^2$.

Similarly, the term that must be added to x^2-8x to make an exact square is $(-\frac{8}{2})^2$, that is 16.

For $x^2-8x+16=(x-4)^2$.

Example 7. Solve $x^2+6x=187.$

To complete the square whose first two terms are x^2+6x, we must add the square of half the coefficient of x, that is $(3)^2$.

The equation becomes

$$x^2+6x+(3)^2=187+(3)^2 \quad . \quad . \quad . \quad \text{(A)}$$

i.e. $x^2+6x+9=187+9=196,$

i.e. $(x+3)^2=196.$

Taking the square root of both sides of the equation,

$$x+3=\pm 14 \quad . \quad . \quad . \quad . \quad \text{(B)}$$

We now have two simple equations:

$$x+3=+14, \text{ and } x+3=-14.$$

Hence the solution is $x=11$, or $x=-17$.

Note (1).—In order to complete the square in line (A), it was necessary to add $(3)^2$ to the L.H.S. If, however, this had been added without changing the R.H.S., the equation would have been altered. **It must be carefully noted that the quantity which is added to the L.H.S., to complete the square, must also be added to the R.H.S.**

Note (2).—In line (B), the square root of 196 was written as ± 14. In the same way, the square root of $(x+3)^2$ is $\pm(x+3)$. It will be seen, however, that the equation $-(x+3)=-14$ is equivalent to $(x+3)=+14$, and that the equation $-(x+3)=14$ is equivalent to $(x+3)=-14$. Thus nothing is gained by putting the dual sign on both sides of the equation.

Example 8. Solve $x^2-65=8x.$

The first step is to write the terms involving x^2 and x on the L.H.S., and the constant term on the R.H.S.

Thus, $x^2-8x=65.$

Completing the square,

$$x^2-8x+(4)^2=65+(4)^2,$$

i.e. $(x-4)^2=81.$

Taking the square root of both sides of the equation,

$$x-4=\pm 9,$$

i.e. $x-4=+9$, or $x-4=-9$.

Hence the solution is $x=13$, or -5.

The questions in Exercise 21c, on page 308, should now be attempted.

When the Coefficient of x^2 is not Unity

151. Example 9. Solve $3x^2+4x=15$.

We cannot complete the square in this case by adding the square of half the coefficient of x, since the coefficient of x^2 is not unity. The first step is to make the coefficient of x^2 unity by dividing *both* sides of the equation by 3.

We obtain $x^2+\frac{4}{3}x=\frac{15}{3}=5$.

Completing the square,

$$x^2+\tfrac{4}{3}x+(\tfrac{2}{3})^2=5+(\tfrac{2}{3})^2=5+\tfrac{4}{9}.$$

i.e. $(x+\frac{2}{3})^2=\frac{49}{9}$.

Taking the square root of both sides,

$$x+\tfrac{2}{3}=\pm\tfrac{7}{3}.$$

i.e. $x+\frac{2}{3}=\frac{7}{3}$, or $x+\frac{2}{3}=-\frac{7}{3}$.

Hence the solution is $x=+\frac{5}{3}$, or -3.

Example 10. Solve $\dfrac{x+1}{x+3}+\dfrac{x+2}{x+4}=\dfrac{2x-3}{x-2}$.

Clearing of fractions by multiplying throughout by

$$(x+3)(x+4)(x-2),$$

and cancelling, we obtain

$$(x+1)(x+4)(x-2)+(x+2)(x+3)(x-2)=(2x-3)(x+3)(x+4).$$

Multiplying out the brackets,

$$x^3+3x^2-6x-8+x^3+3x^2-4x-12=2x^3+11x^2+3x-36.$$

Whence, $5x^2+13x-16=0$.

The first step is to transpose the constant term to the R.H.S.,

$$5x^2+13x=16.$$

Dividing both sides by the coefficient of x^2, that is by 5,

$$x^2+\frac{13x}{5}=\frac{16}{5}.$$

Completing the square, $x^2+\dfrac{13x}{5}+\left(\dfrac{13}{10}\right)^2=\dfrac{16}{5}+\left(\dfrac{13}{10}\right)^2$,

i.e. $\left(x+\dfrac{13}{10}\right)^2=\dfrac{16}{5}+\dfrac{169}{100}=\dfrac{489}{100}$.

Taking square roots, $\quad x+\dfrac{13}{10}=\pm\sqrt{\dfrac{489}{100}}=\pm\dfrac{\sqrt{489}}{10}$,

i.e. $\quad x+\dfrac{13}{10}=\dfrac{\sqrt{489}}{10}$, or $x+\dfrac{13}{10}=-\dfrac{\sqrt{489}}{10}$.

$\therefore \quad x=\dfrac{-13+\sqrt{489}}{10}$, or $\dfrac{-13-\sqrt{489}}{10}$.

Since 489 is not a perfect square, **the only exact values of x which satisfy the equation are those given above.** The two roots are usually written together thus :

$$x=\frac{-13\pm\sqrt{489}}{10}.$$

If, however, an approximation to the value of x is required, say to three places of decimals, then the result can be simplified. For $\sqrt{489}=22{\cdot}11$, correct to two places of decimals.

Hence $x=\dfrac{-13\pm22{\cdot}11}{10}$

$=\dfrac{9{\cdot}11}{10}$, or $\dfrac{-35{\cdot}11}{10}$

$={\cdot}911$, or $-3{\cdot}511$, correct to three places of decimals.

Note.—A square root such as $\sqrt{489}$, which is not exact, is called an **irrational quantity.** When the solution of a quadratic equation contains an irrational quantity, it is impossible to solve it by factorising the quadratic expression as in § 147.

Example 11. Solve the equation $\quad \dfrac{5x-2}{21x^2+x-10}-\dfrac{3x+4}{28x^2-x-15}$

$$=\frac{x-10}{12x^2-17x+6},$$

giving the values of x correct to two decimal places.

The first step is to clear of fractions by multiplying throughout by the L.C.M. of the denominators. To find the L.C.M. it is necessary to factorise each denominator.

The denominators become $(7x+5)(3x-2)$, $(7x+5)(4x-3)$ and $(4x-3)(3x-2)$ respectively.

The L.C.M. is thus $(7x+5)(3x-2)(4x-3)$.

The equation becomes:

$$\frac{(5x-2)(7x+5)(3x-2)(4x-3)}{(7x+5)(3x-2)}-\frac{(3x+4)(7x+5)(3x-2)(4x-3)}{(7x+5)(4x-3)}$$
$$=\frac{(x-10)(7x+5)(3x-2)(4x-3)}{(4x-3)(3x-2)}$$

i.e. $(5x-2)(4x-3)-(3x+4)(3x-2)=(x-10)(7x+5)$.

Multiplying out the brackets,

$$20x^2-23x+6-9x^2-6x+8=7x^2-65x-50.$$

Transposing, $\qquad 4x^2+36x=-64.$

Dividing throughout by 4, $\qquad x^2+9x=-16.$

Completing the square, $\quad x^2+9x+\left(\frac{9}{2}\right)^2=-16+\left(\frac{9}{2}\right)^2,$

$$\text{i.e.}\quad \left(x+\frac{9}{2}\right)^2=-16+\frac{81}{4}=\frac{17}{4}.$$

Taking the square roots, $\qquad x+\frac{9}{2}=\pm\frac{\sqrt{17}}{2}.$

Hence, $\qquad x=\frac{-9\pm\sqrt{17}}{2}.$

Now, $\qquad \sqrt{17}=4{\cdot}12$, correct to two decimal places.

$$\therefore\quad x=\frac{-9\pm4{\cdot}12}{2}=-\frac{4{\cdot}88}{2}, \text{ or } -\frac{13{\cdot}12}{2}$$
$$=-2{\cdot}44, \text{ or } -6{\cdot}56.$$

152. The pupil will see that solving a quadratic equation by the process of completing the square requires the following steps:

1. **Simplify the equation, and arrange it with the terms in x^2 and x on the L.H.S. and the constant on the R.H.S.**

2. **Divide throughout by the coefficient of x^2.**

3. **Complete the square on the L.H.S., by adding to it the square of half the coefficient of x, and add the same quantity to the R.H.S.**
4. **Take the square root of each side, and solve the two simple equations thus obtained.**

The questions in Exercise 21d, page 308, should now be attempted.

Solving by Formula

153. By giving appropriate values to a, b, and c, the equation $ax^2+bx+c=0$ includes all quadratic equations.

We shall now solve the general quadratic equation

$$ax^2+bx+c=0.$$

Transposing the constant term to the R.H.S.,

$$ax^2+bx=-c.$$

Dividing by the coefficient of x^2, that is by a,

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

Completing the square,

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=-\frac{c}{a}+\left(\frac{b}{2a}\right)^2,$$

i.e.

$$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$$

$$=\frac{b^2-4ac}{4a^2} \quad . \quad \text{(A)}$$

Taking the square root of both sides,

$$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$$

$$=\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

Hence,

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a},$$

i.e.

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

Note.—In line A, the terms $-4ac$ and b^2 in the numerators on the R.H.S. have been rearranged, as it is more convenient to commence with a positive term.

The coefficients a, b, c above may have any values whatsoever. Hence the solution will serve as a formula giving the roots of any quadratic equation.

Example 12. Solve the equation $3x^2+7x-26=0$.

Treating this as an instance of the general quadratic equation $ax^2+bx+c=0$, we see that $a=3$, $b=7$, $c=-26$.

Applying the formula, $$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a},$$

$$x=\frac{-7\pm\sqrt{7^2-4\times3\times(-26)}}{2\times3}$$

$$=\frac{-7\pm\sqrt{49+312}}{6}$$

$$=\frac{-7\pm\sqrt{361}}{6}=\frac{-7\pm19}{6}$$

$$=\frac{12}{6}, \text{ or } -\frac{26}{6}=2, \text{ or } -\frac{13}{3}.$$

Example 13. Solve the equation $2x^2-5x-4=0$.

In this example $a=2$, $b=-5$, $c=-4$.

Applying the formula,

$$x=\frac{-(-5)\pm\sqrt{(-5)^2-4\times2\times(-4)}}{2\times2}$$

$$=\frac{5\pm\sqrt{25+32}}{4}=\frac{5\pm\sqrt{57}}{4}.$$

This is the exact solution of the equation, which can now be evaluated to any degree of accuracy required.

154. Example 14. Solve $\quad \frac{5x}{x-1}+\frac{4}{(x-1)^2}=1.$

Before the formula can be applied, the equation must be simplified and written in the standard from $ax^2+bx+c=0$.

Multiplying throughout by the L.C.M. of the denominators, that is by $(x-1)^2$, we obtain

$$5x(x-1)+4=(x-1)^2,$$

which becomes $\quad 4x^2-3x+3=0.$

In this example $a=4$, $b=-3$, $c=3$.

Applying the formula,

$$x=\frac{-(-3)\pm\sqrt{(-3)^2-4\times4\times3}}{2\times4}$$

$$=\frac{3\pm\sqrt{9-48}}{8}=\frac{3\pm\sqrt{-39}}{8}.$$

In § 141 it was pointed out that the square root of a negative quantity is imaginary. Thus $\sqrt{-39}$ cannot be given a numerical value, either exactly or approximately, and so the roots of the equation are imaginary.

The pupil should not confuse an imaginary quantity with an irrational quantity, such as $\sqrt{489}$ obtained in Example 10, which can be evaluated to any required degree of accuracy.

The Discriminant

155. We have shown that the solution of the quadratic equation $ax^2+bx+c=0$ is given by the formula $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. In the remarks that follow it is assumed that a, b, and c are all rational. Hence the only part of the solution which can be other than rational is $\sqrt{b^2-4ac}$. The expression b^2-4ac is called the **discriminant** of the equation, and is denoted by D.

1. **If D is a perfect square, the roots of the equation are rational and unequal.**

2. **If D is positive but not a perfect square, the roots of the equation are irrational and unequal.**

3. **If D is zero, the roots of the equation are $\dfrac{-b\pm 0}{2a}$, and are thus equal, each being $-\dfrac{b}{2a}$.**

4. **If D is negative, the roots of the equation are imaginary.**

The questions in Exercise 21e, page 309, should now be attempted.

156. Three methods of solving a quadratic equation have been explained:

(1) by factorising the quadratic expression,

(2) by completing the square,

(3) by use of the formula.

If the quadratic expression is readily factorisable, method (1) should be employed; if not, then one of the remaining methods.

Example 15. What values of x will make $3x^2-10a^2$ equal to $13ax$?

We have to solve the equation,

$$3x^2-10a^2=13ax.$$

Writing this as a quadratic in x,

$$3x^2-13ax-10a^2=0.$$

The factors are obtained from $\dfrac{1}{3}\Big|\dfrac{-5a}{2a}$

$$\therefore\quad (x-5a)(3x+2a)=0.$$

Hence, $\quad x=5a$, or $-\frac{2}{3}a$.

Example 16. Solve $\quad 35x^2+(3a+46b)x-(2a^2-ab-15b^2)=0.$

We first factorise the constant term:

$$2a^2-ab-15b^2=(a-3b)(2a+5b).$$

The factors of the quadratic expression in x are obtained from

$$\frac{5}{7}\Big|\frac{-(a-3b)}{2a+5b}$$

$$\therefore\quad [5x-(a-3b)][7x+(2a+5b)]=0.$$

Hence, $\quad x=\dfrac{a-3b}{5}$, or $-\dfrac{2a+5b}{7}$.

Example 17. Find the value of the fraction $\frac{x}{y}$ given that

$$24x^2+xy-10y^2=0.$$

Dividing throughout by y^2, we obtain

$$24\frac{x^2}{y^2}+\frac{x}{y}-10=0.$$

This is a quadratic equation in the unknown $\frac{x}{y}$, and can be solved by factorising.

Thus, $$\left(8\frac{x}{y}-5\right)\left(3\frac{x}{y}+2\right)=0.$$

Hence, $$\frac{x}{y}=\frac{5}{8}, \text{ or } -\frac{2}{3}.$$

Example 18. Solve $(6x^2-x)^2-17(6x^2-x)+60=0.$

This equation, if the brackets are multiplied out, is of the fourth degree in x. It is, however, of the second degree in the unknown $(6x^2-x)$, which we may denote by y. The equation then becomes $$y^2-17y+60=0,$$

i.e. $$(y-5)(y-12)=0.$$

Hence, $$y=5, \text{ or } 12.$$

Now, $$y=6x^2-x.$$

We thus have two quadratic equations in x to solve, namely,

$$6x^2-x=5, \text{ and } 6x^2-x=12.$$

i.e. $$6x^2-x-5=0, \text{ and } 6x^2-x-12=0.$$

i.e. $$(x-1)(6x+5)=0, \text{ and } (2x-3)(3x+4)=0.$$

$$\therefore \quad x=1, -\tfrac{5}{6}, \tfrac{3}{2}, \text{ or } -\tfrac{4}{3},$$

which are the four roots of the original equation of the fourth degree.

The questions in Exercise 21f, page 311, should now be attempted.

Exercise 21a

Solve by factorising, and verify your solutions:

1. $x^2-5x+6=0.$ **2.** $x^2-4x+3=0.$ **3.** $x^2-14x+13=0.$
4. $x^2-7x+12=0.$ **5.** $x^2-19x+88=0.$ **6.** $x^2+4x+3=0.$
7. $x^2+10x+21=0.$ **8.** $x^2+9x+20=0.$ **9.** $x^2+6x+9=0.$

10. $-x^2+x+6=0.$
11. $x^2-3x-10=0.$
12. $x^2-12x-45=0.$
13. $-x^2-12x+28=0.$
14. $x^2+10x-39=0.$
15. $x^2+2x-15=0.$
16. $-x^2-x+30=0.$
17. $x^2-4x=0.$
18. $x^2-16=0.$
19. $x^2-25=0.$
20. $x^2=9.$
21. $-2x^2=-8.$
22. $x^2=81.$
23. $(x+1)^2-36=0.$
24. $(x+2)^2=49.$
25. $(x-5)^2-64=0.$
26. $(x-2)^2=1.$
27. $8x=20-x^2.$
28. $x^2=x+12.$
29. $x^2-1=x+1.$
30. $x^2+143=24x.$
31. $x^2+50=15x-6.$
32. $4x^2-1=0.$
33. $25x^2=121.$
34. $x^2-a^2=0.$
35. $x^2=7ax.$
36. $(x-3)^2-9=0.$
37. $x^2-2bx=0.$
38. $x^2+3x=0.$
39. $x^2=-ax.$
40. $6x^2-11x+4=0.$
41. $10x^2-21x+9=0.$
42. $2x^2+5x-3=0.$
43. $18x=16-13x^2.$
44. $9x^2-13x-10=0.$
45. $10x^2=9x+9.$
46. $11x^2+24x=-4.$
47. $12x^2+32x+21=0.$
48. $11x^2=4x+7.$
49. $1-x^2=-8.$
50. $8x-3x^2=5.$
51. $x^2-6ax+9a^2=0.$
52. $x^2=2bx-b^2.$
53. $7x^2-21x=0.$
54. $7a^2=12ax-5x^2.$
55. $18x^2=13bx+11b^2.$
56. $3a^2-4ax-4x^2=0.$
57. $(x+3)^2-b^2=0.$
58. $4x^2-(x+1)^2=0.$
59. $x-10=6x(2-x).$

Exercise 21b

Solve the equations :

1. $(1-x)(x+2)=(1-x).$
2. $(2x+3)(x+1)=(2x+3).$
3. $2x(4x-3)=x.$
4. $ax(x+3)=2ax.$
5. $(x-a)(x-1)=(x-a).$
6. $(x+a)(x+1)=2(x+a).$
7. $(x-4)^2=(x-4).$
8. $(x+2)^2=2(x+2).$
9. $x^2-25=(x+5).$
10. $8x^2-10x+3=2x^2+3x-3.$
11. $17x^2+11x-5=5x^2-6x+2.$
12. $(x-3)(11x^2+15x+4)=(x-3)(x^2-2x+1).$
13. $(2x-1)(5x^2-17)=(2x-1)(x^2-1).$
14. $(3x+2)(2x^2-7)=(3x+2)(3x+2).$
15. $(3x-1)(2x+7)=(5x+2)(1-2x).$
16. $(x+4)(6x+1)=-x(10x+9).$
17. $(2x+1)(x+3)=x^2+4x+1.$
18. $(5x+1)(2x+3)=(1-x)(1+3x).$
19. $x^2+\frac{11}{4}x+\frac{5}{8}=0.$
20. $x^2+2=\frac{11}{3}x.$
21. $x+\dfrac{1}{x}=2.$
22. $x-\dfrac{1}{x}=0.$
23. $x+2=\dfrac{1}{x+2}.$
24. $\dfrac{x}{2}-\dfrac{1}{x-1}=0.$

25. $\frac{x}{x-1}+\frac{2}{x-1}=2x.$

26. $\frac{2x+1}{x-1}-1=\frac{3x+2}{x+3}.$

27. $\frac{3x-2}{x+2}-\frac{x-2}{2x+1}=\frac{2}{3}.$

28. $\frac{5x-1}{4x-3}-\frac{2x-1}{x-3}=0.$

29. $\frac{2x}{x+3}-\frac{2x}{3x-1}=-1\frac{5}{7}.$

30. $\frac{5x+1}{2x}+\frac{2x}{x-5}=-\frac{1}{x}.$

31. $\frac{1}{x}-2+\frac{3}{x^2}=0.$

32. $\frac{x}{2}-\frac{2}{5x}=\cdot 8.$

33. $\frac{3}{8x}-3\cdot 1=-4\cdot 7x.$

34. $(x-\frac{6}{7})(x-\frac{2}{3})=\frac{10}{21}(x-\frac{1}{2})(x-\frac{4}{5}).$

35. $\frac{5x}{\cdot 9}-3\frac{1}{27}=\frac{4}{9x}.$

36. $\frac{(x+\frac{3}{2})}{(3x-2)}-\frac{(x-\frac{3}{4})}{(x-\frac{1}{2})}=-\frac{9}{4}.$

37. $\frac{1}{x-1}-\frac{2}{x}=\frac{3}{x+1}-\frac{4}{x+2}.$

38. $\frac{7}{x+4}-\frac{4}{x+1}=\frac{5}{x+2}-\frac{2}{x-1}.$

39. $\frac{3}{x-3}-\frac{8}{x+2}=\frac{1}{x-4}-\frac{6}{x+6}.$

40. $\frac{3x-2}{2x-3}-\frac{7-4x}{4x-3}=\frac{7}{2}.$

41. $\frac{2(2x-1)}{2x-3}+\frac{6x-5}{2x+3}=\frac{10x-27}{2x-5}.$

42. $\frac{1}{(6x-1)(15x^2-22x-5)}=\frac{2}{(5x-2)(24x^2+50x-9)}.$

43. $\frac{x-1}{x+2}-\frac{1}{4-x^2}=\frac{3}{5}.$

44. $\frac{3x+5}{2x+3}+\frac{4-x}{4x^2-9}=\frac{11}{9}.$

45. $\frac{2x-1}{x+1}+\frac{x-2}{x+2}=\frac{3x-1}{x+3}.$

46. $2ax^2+(a^2b-2)x-ab=0.$

47. $3ax^2-(2a^2+3)x+2a=0.$

48. $\frac{b}{a}x^2+\frac{a}{b}=2x.$

49. $6a^2bx^2+(3ab^2-2a)x=b.$

50. $\frac{x}{a+1}+\frac{3}{(a-1)x}=-\frac{a^2+2}{a^2-1}.$

51. $\frac{3}{3x-2a}-\frac{2}{x}=\frac{9}{3x+2a}-\frac{12}{3x+4a}.$

52. $9x^2-3\left(\frac{a-b}{ab}\right)x=\frac{1}{ab}.$

53. $\frac{x}{b}+\frac{b}{10x}=-\frac{2a^2+5}{10a}.$

54. $\frac{1}{x+a}+\frac{3}{4x+a}=\frac{2}{2x-a}.$

55. $\frac{a}{3x-a}-\frac{2a}{5x+a}=\frac{1}{6}.$

Exercise 21c

Solve the following equations by completing the square :

1. $x^2+2x=15.$
2. $x^2+10x=39.$
3. $x^2+6x=27.$
4. $x^2+3x=40.$
5. $x^2+5x=36.$
6. $x^2+12x=28.$
7. $x^2-4x=77.$
8. $x^2-2x=35.$
9. $x^2+7x=8.$
10. $x^2-3x=10.$
11. $x^2+8x=-15.$
12. $x^2+16x=-63.$
13. $x^2-9x=70.$
14. $x^2+13x=14.$
15. $x^2+13x=-30.$
16. $x^2-5x=-6.$
17. $x^2-14x=-48.$
18. $x^2+9x=-20.$
19. $x^2-x=6.$
20. $x^2-19x=-88.$
21. $x^2+x=12.$
22. $x^2-44=7x.$
23. $x^2-39=10x.$
24. $x^2+21=10x.$
25. $2x+35=x^2.$
26. $45-4x=x^2.$
27. $x^2+18=9x.$
28. $17x=x^2+52.$
29. $12x=11+x^2.$
30. $x^2=x+56.$
31. $x^2+2bx=3b^2.$
32. $x^2+bx=6b^2.$
33. $x^2-4ax=5a^2.$
34. $x^2-7cx=-10c^2.$
35. $x^2-7a^2=6ax.$
36. $x^2+16b^2=-10bx.$
37. $x^2+\frac{2}{3}x=7.$
38. $x^2+\frac{1}{2}x=3.$
39. $x^2-\frac{4}{5}x=\frac{1}{5}.$
40. $x^2+\frac{5}{3}x=\frac{2}{3}.$
41. $x^2-\frac{3}{2}x=-\frac{1}{2}.$
42. $x^2+1\frac{1}{4}x=-\frac{3}{2}.$
43. $x^2-\frac{x}{6}=1\frac{1}{6}.$
44. $x^2+\frac{2ax}{7}=\frac{5}{7}a^2.$
45. $-x^2+\frac{x}{4}=-\frac{3}{8}.$

Exercise 21d

Solve the following equations by completing the square :

1. $2x^2+5x=3.$
2. $3x^2+10x=8.$
3. $7x^2-5x=2.$
4. $4x^2=15-7x.$
5. $13x+3=10x^2.$
6. $3x^2-13x=-12.$
7. $5x^2+8=14x.$
8. $9x^2+18x=-8.$
9. $4x^2+15=-16x.$
10. $6x^2+5x-6=0.$
11. $9(2x-1)=8x^2.$
12. $6x^2=65+29x.$
13. $14x^2+11x=-2.$
14. $20+9x=20x^2.$

Solve the following equations, and give the value of irrational roots correct to two decimal places :

15. $x^2+2x=1$.

16. $x^2+3x=5$.

17. $x^2-10x=3$.

18. $x^2-7x=6$.

19. $x^2+7x=+13$.

20. $x^2-4x=-2$.

21. $x^2+8=7x$.

22. $x^2=5x+11$.

23. $x^2+4=-6x$.

24. $x^2+11x-9=0$.

25. $x^2=14x+8$.

26. $x^2-2=-8x$.

27. $2x^2+6x=3$.

28. $3x^2+2x=6$.

29. $6x^2-10x=13$.

30. $3x^2-7x=16$.

31. $4x+5=7x^2$.

32. $4x^2=1-5x$.

33. $10x^2-20x=-3$.

34. $5x^2-13x=-5$.

35. $11x^2=30x+20$.

36. $12x^2+25x=-10$.

37. $4x^2+3=-9x$.

38. $9x^2+3x-4=0$.

39. $5x+7-8x^2=0$.

40. $35-41x=-9x^2$.

41. $6x^2+27x=-22$.

42. $3x^2+17+16x=0$.

43. $\dfrac{x+8}{x+1}=\dfrac{5}{x}$.

44. $(2x+15)(2x-1)=10(2x+1)$.

45. $\dfrac{2x+3}{x-2}=\dfrac{3x+4}{4x-1}$.

46. $\dfrac{1}{2x^2}+\dfrac{25}{4x}=2$.

47. $(5-2x)(3x+4)=(7x+1)(2x-3)$.

48. $(4x-3)(3x+2)=(4x+9)(5x-1)$.

49. $\dfrac{4x-3}{2x+1}+\dfrac{2x-3}{2x+3}=\dfrac{6x-1}{2x+5}$.

50. $\dfrac{2x+1}{x-1}-\dfrac{7x+2}{3x-2}=\dfrac{2-x}{3x-1}$.

51. $\dfrac{1+2x}{x^2+x-12}+\dfrac{3x-1}{x^2-2x-3}=\dfrac{2x+3}{x^2+5x+4}$.

52. $\dfrac{7x+1}{10x^2+x-3}-\dfrac{4x-3}{8x^2-18x+7}=\dfrac{6x-5}{20x^2-23x-21}$.

53. $\dfrac{x}{6x^2-x-15}-\dfrac{3}{6x^2-19x+15}=\dfrac{1-3x}{4x^2-9}$.

Exercise 21e

Use the formula to solve the following equations, giving your solutions correct to two decimal places :

1. $x^2+5x+3=0$.

2. $4x^2+8x-1=0$.

3. $3x^2-11x-5=0$.

4. $10x^2-13x+2=0$.

5. $-7x^2+3x+7=0$.

6. $22x=11x^2+10$.

7. $1-x^2=4x$.

8. $x^2+\frac{5}{3}x+\frac{1}{2}=0$.

9. $\frac{2}{3}x^2+x=\frac{2}{21}$.

10. $\frac{7x}{3}-\frac{6}{3x}-3=0$.

11. $\frac{3x}{2-x}=\frac{1}{4-x^2}$.

12. $\cdot 03-\frac{\cdot 17}{x}+\frac{\cdot 21}{x^2}=0$.

13. $\frac{x}{3}\left(2x-3+\frac{3}{4x}\right)=0$.

14. $7x+1\cdot 3=3\cdot 5x^2$.

Find the solutions of the following quadratic equations in x:

15. $x^2+5ax-7a^2=0$.

16. $p^2x^2-3px+1=0$.

17. $\frac{x}{b}+\frac{b}{a}=\frac{b}{x}$.

18. $x-a+\frac{1}{x-a}=2$.

19. $3(x-b)^2+2(b-x)-3=0$.

20. $x(cx-2)=cx-1$.

21. $\frac{m-1}{m+1}x+\frac{1}{x}=-\frac{2m+1}{m+1}$.

22. $x^2+2ax+a^2=a+b$.

23. $\frac{p}{q}x^2+\frac{q}{p}=1-2x$.

In the examples which follow, use the discriminant to examine the roots of the equations.

24. State in which equations the roots are real, and in which imaginary.

(a) $x^2+5x+1=0$. (b) $5x^2=4x+2$.
(c) $2x^2+7=6x$. (d) $x^2+3x+2=0$.

25. Which of the following equations have irrational roots, and which rational roots?

(a) $7x^2+3x-5=0$. (b) $3x^2+8x-3=0$.
(c) $5x^2+7x-6=0$. (d) $4x^2-13x=5$.

26. Which of the following equations have imaginary roots, and which rational roots?

(a) $2x^2-5x+4=0$. (b) $6x^2-12x+7=0$.
(c) $21x^2+22x-8=0$. (d) $9x^2+6x+2=0$.

27. Which of the following equations in x have real, which irrational, and which imaginary roots?

(a) $(mx)^2+3mx+1=0$. (b) $ax^2+bx+\frac{b^2}{a}=0$.

(c) $px^2+q=(p+q)x$.

Exercise 21f

Solve the following equations (1 to 5) by any suitable method:

1. $2(x-5)^2+7(x-5)+3=0$.
2. $5(x-1)^2=7(x-1)+6$.
3. $1+\frac{3}{x+2}=\frac{10}{(x+2)^2}$.
4. $3(2x-1)^2-2x+1=2$.
5. $3x+2+\frac{3}{3x+2}=5$.
6. Find the values of $\frac{x}{y}$ from the equation,
$$\frac{6}{y^2}=\frac{20}{x^2}-\frac{7}{xy}.$$
7. Find the values of $\frac{x}{y}$ in terms of a and b, given that
$$\frac{by}{ax}-\frac{x}{y}=\frac{1}{a}-b.$$
8. Find the values of $\frac{x}{y}$, given that
$$6\left(\frac{x}{y}-1\right)^2-5\left(\frac{x}{y}-1\right)=4.$$
9. Find the values of x in terms of y, given that
$$(2x-y)^2=2(x+2)-y.$$
10. Find the values of x in terms of y, given
$$3(x-3y)^2=(3y-x+4).$$
11. Solve the equation $(a+b)x^2-(b+a)x=b-a$.
12. Solve the equation in x,
$$x+\frac{a(a-2)-x}{x+2a}=\frac{b}{x+2a}-1.$$
13. Solve $(x+a)^2+(x+b)^2=(a-b)^2$.
14. Solve $ac(x+\frac{1}{2})^2-(2a-3c)(x+\frac{1}{2})-6=0$.
15. Solve $4x^4-17x^2+4=0$.
16. Find the values of the product xy, given $15x^2y^2+11xy=14$.
17. Find the values of x which satisfy the equation,
$$(x^2+2x)^2-2(x^2+2x)-3=0.$$

18. Solve the equation,

$$\left(x+\frac{6}{x}\right)^2+2\left(x+\frac{6}{x}\right)=35.$$

19. Solve the equation,

$$2\left(\frac{1}{x-1}\right)^2-11\left(\frac{1}{x-1}\right)+15=0.$$

20. Given that $x=\frac{1}{2}$ is a root of the equation $2x^3+5x^2+x-2=0$, find the other roots.

21. Given $x=3$ is a root of the equation $3x^3=2x^2+18x+9$, find the other roots correct to two decimal places.

22. Find one of the roots of the equation $2x^3-9x^2+7x+6=0$ by use of the Factor Theorem; then solve the equation completely.

CHAPTER XXII

Problems Leading to Quadratic Equations

157. We shall now solve some problems which lead to a quadratic equation in the unknown.

Example 1. Divide 20 into two parts such that the sum of their squares is 250.

Denote one of the parts by x, then the other part is $(20-x)$.

The squares of the parts are thus x^2 and $(20-x)^2$.

Hence the conditions of the problem give

$$x^2+(20-x)^2=250,$$

i.e. $$x^2+400-40x+x^2=250.$$

Transposing, $$2x^2-40x+150=0.$$

Dividing throughout by 2,

$$x^2-20x+75=0.$$

$$\therefore \quad (x-5)(x-15)=0.$$

Hence, $$x=5, \text{ or } x=15.$$

If x, that is the first part, is 5, then the second part, which is $20-x$, is 15.

If x, that is the first part, is 15, then the second part, which is $20-x$, is 5.

It is seen that both roots of the quadratic equation give the same solution to the problem. The parts are thus 5 and 15.

Note.—The pupil should make a point of verifying all solutions to problems. In the case of this problem

$$5^2+15^2=25+225=250,$$

which verifies the solution.

Example 2. A and B each walk 24 miles. The sum of their speeds is 7 m.p.h., and the sum of the times taken is 14 hours. Find their speeds.

Denote the speed of A by x m.p.h.; then B's speed is $(7-x)$ m.p.h.

The time A takes to walk 24 miles is $\frac{24}{x}$ hours.

The time B takes to walk 24 miles is $\frac{24}{7-x}$ hours.

Hence the condition of the problem gives

$$\frac{24}{x}+\frac{24}{7-x}=14.$$

Clearing of fractions,

$$24(7-x)+24x=14x(7-x),$$

i.e. $$168-24x+24x=98x-14x^2.$$

Transposing, $$14x^2-98x+168=0.$$

Dividing throughout by 14,

$$x^2-7x+12=0.$$

$$\therefore \quad (x-3)(x-4)=0.$$

Hence, $$x=3, \text{ or } 4.$$

If A's speed, that is x, is 3 m.p.h., then B's speed, that is $7-x$, is 4 m.p.h.

If A's speed, that is x, is 4 m.p.h., then B's speed, that is $7-x$, is 3 m.p.h.

Thus their speeds are 3 m.p.h. and 4 m.p.h.

Note.—As the conditions of the question do not differentiate between A and B, we should expect an answer in the dual form, as obtained from the solution of the quadratic equation.

The questions in Exercise 22a, page 320, should now be attempted.

The Inadmissible Root

158. In the previous examples, each of the roots of the quadratic equation gives an admissible answer to the problem. It often happens, however, that one of the roots of the equation cannot be intelligibly applied to the conditions of the question. For example, we may be asked to find the digits of a number, and one of the roots may be fractional, or negative. Such a root must of course be rejected, and the remaining root gives the solution.

Example 3. Two cubes differ in volume by 11 c. ft. 1305 c. ins. If their edges differ by 1 ft., find their dimensions.

As has been pointed out in Part I (§ 48, Example 5), it is essential that the quantities involved in a problem be denoted in terms of the same unit.

Working in inches:

$$1 \text{ c. ft.} = (12)^3 \text{ c. ins.} = 1728 \text{ c. ins.}$$
$$\therefore \quad 11 \text{ c. ft. } 1305 \text{ c. ins.} = (11 \times 1728 + 1305) \text{ c. ins.}$$
$$= 20313 \text{ c. ins.}$$

Denote the edge of the smaller cube by x inches, then the edge of the larger cube is $(x+12)$ inches.

The volumes of the cubes are x^3 and $(x+12)^3$ c. ins. respectively. Hence the condition of the problem gives

$$(x+12)^3 - x^3 = 20313,$$
i.e. $$x^3 + 36x^2 + 432x + 1728 - x^3 = 20313.$$
Transposing, $$36x^2 + 432x - 18585 = 0.$$
Dividing throughout by 9, $4x^2 + 48x - 2065 = 0,$
$$\therefore \quad (2x-35)(2x+59) = 0.$$
Hence, $$x = 17\tfrac{1}{2}, \text{ or } -29\tfrac{1}{2}.$$

As x denoted the length of the edge of the first cube, the root $x = -29\frac{1}{2}$ has no intelligible meaning.

Thus, $x = 17\frac{1}{2}$ gives the edge of the smaller cube.

The edge of the larger cube is $(x+12)$, that is $29\frac{1}{2}$ inches.

Hence the cubes have edges of $17\frac{1}{2}$ and $29\frac{1}{2}$ inches respectively.

The pupil should verify that $(29\frac{1}{2})^3 - (17\frac{1}{2})^3 = 20313$.

Interpretation of the Negative Root

159. **Example 4.** If the price of eggs is reduced 3d. a dozen, four more can be purchased for half a crown. Find the original price of a dozen eggs.

Working in pence:

Denote the original price of a dozen eggs by x pence.

Therefore one egg costs $\dfrac{x}{12}$ pence.

Hence the number of eggs purchased for half a crown is $30 \div \dfrac{x}{12}$, that is $\dfrac{360}{x}$.

The new price of a dozen eggs is $(x-3)$ pence; thus the new price of one egg is $\frac{x-3}{12}$ pence.

Hence the number of eggs bought for half a crown is

$$30 \div \frac{x-3}{12}, \text{ that is } \frac{360}{x-3}.$$

Hence the condition of the problem gives

$$\frac{360}{x-3} = \frac{360}{x} + 4.$$

Clearing of fractions, $\quad 360x = 360(x-3) + 4x(x-3)$,

i.e. $\quad 360x = 360x - 1080 + 4x^2 - 12x$.

Transposing, $4x^2 - 12x - 1080 = 0$.

Dividing throughout by 4,

$$x^2 - 3x - 270 = 0,$$

$$\therefore \quad (x-18)(x+15) = 0.$$

Hence, $\quad x = 18$, or -15.

Rejecting the root $x = -15$, which is inadmissible,

$$x = 18.$$

Thus the original price of a dozen eggs is 18 pence, that is 1s. 6d.

Note.—Although the root $x = -15$ has been rejected, it should be possible to give it a *theoretical explanation*, as it has occurred as the result of the algebraical working.

If the original price is -15 pence a dozen, then the new price is $-15-3$, that is -18 pence a dozen. The original number bought for 2s. 6d. is

$$30 \div \left(-\tfrac{15}{12}\right) = -\tfrac{360}{15} = -24.$$

The new number bought for 2s. 6d. is

$$30 \div \left(-\tfrac{18}{12}\right) = -\tfrac{360}{18} = -20;$$

and -20 *is* equal to $-24+4$.

Thus the condition of the problem is satisfied *theoretically*, since

$$30 \div \left(-\tfrac{18}{12}\right) = 30 \div \left(-\tfrac{15}{12}\right) + 4.$$

If the signs are changed throughout on both sides, we obtain

$$30\div\frac{18}{12}=30\div\frac{15}{12}-4.$$

This shows that if the original price were 15 pence a dozen, and the price be *increased* by 3d., that is to 18 pence a dozen, four *fewer* would be purchasable for half a crown. Hence the negative root, namely -15, obtained as a solution of the quadratic equation, is the correct answer to the problem obtained by changing the words 'reduced' to 'increased,' and 'more' to 'fewer.' **It will frequently be found that a negative root of a quadratic equation, obtained in solving a problem, can be interpreted as a solution of the problem** by making similar word changes. The student will find it instructive to refer back to Chapter III, on Directed Numbers, for the effect of such word changes.

The questions in Exercise 22b, page 322, should now be attempted.

160. The following examples are rather more difficult.

Example 5. A father is three times as old as his son. The product of the father's age in three years' time, and the son's age three years ago, is twenty times the present age of the son. Find their ages.

Denote the son's present age by x.

Thus the father's present age is $3x$.

In three years' time the father's age will be $3x+3$, while three years ago the son's age was $x-3$.

Hence the condition of the problem gives

$$(3x+3)(x-3)=20x,$$

i.e. $$3x^2-6x-9=20x.$$

Transposing, $$3x^2-26x-9=0,$$

$$\therefore\quad (x-9)(3x+1)=0.$$

Hence, $$x=9, \text{ or } -\tfrac{1}{3}.$$

Rejecting the root $x=-\frac{1}{3}$, which is inadmissible,

$$x=9.$$

Hence the son's age is 9, and the father's age is 27.

Example 6. The expression ax^2+bx+c has the values 9, 21, 43, when x has the values 1, 2, 3 respectively. Find the values of x which will make the expression equal to 15.

If $x=1$, the expression ax^2+bx+c becomes $a+b+c$.
Hence $a+b+c=9$ . . . . (1)
If $x=2$, the expression ax^2+bx+c becomes $4a+2b+c$.
Hence $4a+2b+c=21$ . . . . (2)
If $x=3$, the expression ax^2+bx+c becomes $9a+3b+c$.
Hence $9a+3b+c=43$ . . . . (3)
Solving equations (1), (2) and (3), we obtain $a=5$, $b=-3$, $c=7$.

Thus the expression becomes $5x^2-3x+7$.
Hence the condition of the problem gives

$$5x^2-3x+7=15.$$

Transposing, $$5x^2-3x-8=0,$$

$$\therefore \quad (x+1)(5x-8)=0.$$

Hence the solution is $x=-1$, or $\frac{8}{5}$.

Note.—In this example it will be noticed that both roots of the equation are admissible as satisfying the conditions of the problem.

Example 7. One pipe can fill a cistern in 10 minutes less than another pipe, but takes 10 minutes 25 seconds longer than the two together. Find the time each takes separately to fill the cistern.

Working in minutes:

Denote the time the first pipe takes by x minutes. Then the time the second pipe takes is $(x+10)$ minutes.

The first pipe fills $\frac{1}{x}$ of the cistern in a minute.

The second pipe fills $\frac{1}{x+10}$ of the cistern in a minute.

Hence the two together fill $\left(\frac{1}{x}+\frac{1}{x+10}\right)$ of the cistern in a minute.

But the question states that the two together take $(x-10\frac{5}{12})$ minutes, so that the two together fill $\frac{1}{x-10\frac{5}{12}}$ of the cistern in a minute.

Hence, $$\frac{1}{x}+\frac{1}{x+10}=\frac{1}{x-10\frac{5}{12}},$$

i.e. $$\frac{x+10+x}{x(x+10)}=\frac{12}{12x-125}.$$

Clearing of fractions,

$$(2x+10)(12x-125)=12x(x+10),$$

i.e. $$24x^2-130x-1250=12x^2+120x.$$

Transposing, $$12x^2-250x-1250=0.$$

Dividing throughout by 2,

$$6x^2-125x-625=0,$$

$$\therefore\quad (x-25)(6x+25)=0.$$

Hence, $$x=25, \text{ or } -\frac{25}{6}.$$

Rejecting the negative root, we have $x=25$.

Thus the first pipe takes 25 minutes, and the second pipe takes 35 minutes to fill the cistern.

Example 8. A number consists of three digits, the tens digit being one more than the hundreds digit and one less than the units digit. The number is equal to the product of the units digit and a two-digit number whose tens digit is twice the original hundreds digit and whose units digit is the sum of the original tens and units digits. Find the number.

Denote the hundreds digit by x.

Then the tens digit is $(x+1)$, and the units digit is $(x+2)$.

Hence the value of the number is

$$100x+10(x+1)+x+2=111x+12.$$

This number is the product of the units digit, that is $(x+2)$, and of a number whose tens digit is $2x$ and whose units digit is

$$(x+1)+(x+2), \text{ that is } (2x+3).$$

Hence, $$111x+12=(x+2)[10(2x)+(2x+3)]$$
$$=(x+2)(22x+3),$$

i.e. $$111x+12=22x^2+47x+6.$$

Transposing, $$22x^2-64x-6=0.$$

Dividing throughout by 2,

$$11x^2-32x-3=0,$$

$$\therefore\ (x-3)(11x+1)=0.$$

Hence, $$x=3, \text{ or } -\frac{1}{11}.$$

Rejecting the root $x=-\frac{1}{11}$, which is *not a digit*, we have $x=3$.

Thus the number is 345.

The questions in Exercise 22c, page 325, should now be attempted.

Exercise 22a

1. If the square of a certain number be subtracted from six times that number, the result is 9. Find the number.
2. If four times the square of a certain quantity be subtracted from 28 times that quantity, the result is 49. What is the quantity ?
3. Divide 50 into two parts such that their product is 429.
4. A person walks to the pillar-box, half a mile away, and back in 25 minutes, his rate of walking each way being uniform. If the sum of these two rates is 5 miles per hour, find what each rate is.
5. Two people each walk 21 miles. If the sum of their speeds is $7\frac{1}{2}$ miles per hour, and the sum of their times is 11 hours 36 minutes, find what these times are.
6. A number added to 20 times its reciprocal is equal to 9. What is the number ?
7. The sum of two numbers is 18 and the sum of their squares is 170. Find the numbers.
8. Divide 32 into two parts such that the sum of the squares of the parts is 514.
9. The sum of two numbers is 58, while their product is 777. Find the numbers.
10. A rectangular field has an area of 1500 square yards and its perimeter is 160 yds. Find the lengths of its sides.

11. Find a number whose square added to 63 is equal to 16 times the number.

12. Find a number which when increased by 20 is equal to 9 times the square root of the number.

13. *A* spends 1s. 8d. on packets of chocolates. *B*, who buys packets costing ½d. less, spends 1s., and receives two packets less than *A*. How much does *A* pay for one packet ?

14. A boy walks for 9 miles. After a rest he walks for 5 miles at a rate ½ m.p.h. slower than before, and takes 1 hour less than before he rested. What were his two rates of walking ?

15. A wire 80 inches long is bent into the shape of a rectangle whose area is 364 square inches. Find the dimensions of the rectangle.

16. The hypotenuse of a right-angled triangle is 25 inches and its perimeter is 56 inches. Find the lengths of the other two sides.

17. A stationer spends £6 on a stock of fountain pens. Later he spends another £3 in buying a stock of a different pen costing 6d. less, and receives 16 fewer pens. How many pens did he obtain altogether ?

18. AB is a straight line 17 cms. long. If X is a point on the line such that $AX \,.\, XB = 55$ sq. cms., find the length of AX correct to the nearest tenth of a mm.

19. A man drives to a town 21 miles away. He returns by a route which is 4 miles shorter, but as his speed is 2 m.p.h. less he saves only 5 minutes on the return journey. At what speed did he drive to the town ?

20. A boy who has yet to receive his marks for three subjects, has a total of 372. His marks for the remaining three subjects are 58, 82, 64, which raises his average by 2. Find the number of subjects he took in all.

21. A number of people shared a coach, the charge for which was £3, 11s. 6d. One member of the party paid 8s. 6d. without waiting to find his correct share, with the result that the others paid 3d. less each than they should have done. How many people were there in the party ?

22. A wire 20 cms. long is bent into a rectangle whose diagonal is 7½ cms. in length. Find the length of the sides of the rectangle correct to the nearest tenth of a mm.

Exercise 22b

1. The difference of two numbers is 3, and the sum of their squares is 185; find them.
2. The product of two consecutive even numbers is 168; what are they?
3. Find two consecutive numbers such that the sum of their squares is equal to 613.
4. Two numbers which differ by 4 have a product of 221. Find the numbers.
5. Find two consecutive odd numbers the sum of the squares of which is 290.
6. Two numbers differ by 9, and the square of the larger added to four times the square of the smaller is 452. Find the numbers.
7. Find a number which is greater by unity than 20 times its reciprocal.
8. Find two numbers which differ by 4, and are such that the sum of their reciprocals is $\frac{5}{24}$.
9. Find two numbers which differ by 3, and are such that the difference of their reciprocals is $\frac{1}{18}$.
10. One number is four times another. If each be increased by 3, the sum of their reciprocals is $\frac{4}{27}$. Find the numbers.
11. A field is 11 yds. longer than it is broad, and has an area of 6 acres. Find its length and breadth.
12. A train travelling at x m.p.h. for $(x+5)$ minutes travels $12\frac{1}{2}$ miles. Find x.
13. Two trains both cover a distance of 260 miles, the average speed of one being 2 m.p.h. more than that of the other. It is found that the faster train takes 12 minutes less for the journey. What are the two speeds?
14. A rectangle whose area is 255 sq. ft. has its length and breadth diminished by 1 yard and 1 foot respectively, and thus becomes a square. Find the length of a side of the square.
15. A man walks to his office and back in $3\frac{1}{4}$ hours, his rate on the return journey being $\frac{1}{2}$ m.p.h. slower than on the outward journey. If he lives $5\frac{1}{4}$ miles from his office, find the rates of walking.
16. 99 bundles of wood are divided among a certain number of poor persons; if each had received 2 bundles more, he would

have received as many bundles as there were persons. How many persons were there?

17. In a concert hall 600 persons are seated on benches of equal length. If there were 10 fewer benches, it would be necessary that two persons more should sit on each bench. Find the number of benches.

What problem is suggested by the negative root of the equation?

18. A sum of £17 is divided among a certain number of persons. If each had received 3 shillings more, he would have received as many shillings as there are persons. How many persons are there?

State a problem that can be obtained from the negative root of the equation.

19. Two motor-cars race over a 330 miles' course. One car, running at a speed 5 m.p.h. greater than the other, wins by 55 minutes. Find the two speeds.

20. AB is a straight line 1 foot long. P is a point in AB such that $AB \cdot AP = PB^2$. Find the length of AP correct to the nearest hundredth of an inch.

21. Answer Question 20 if P is a point on AB produced.

22. A fraction, whose numerator is 2 less than its denominator, is doubled if 21 be added to the numerator and 9 be added to the denominator. Find the fraction.

23. A rectangle of sides 13 inches and 9 inches has a strip of uniform width cut off all round it. If this reduces the area by 57 sq. ins., find the width of the strip.

24. How many bars of chocolate can be bought for 2s. 6d. if three more for the money would lower the price sixpence a dozen.

Change the wording to form the problem which interprets the negative root of the equation.

25. Two trains have each a distance of 200 miles to cover. Their speeds differ by 10 m.p.h., and their times by 1 hour. How long does the faster train take?

26. In estimating for a contract, an employer reckons that he will have to pay his labourers £42 a week altogether. Later he finds that he needs 5 more labourers, and if he does not wish to increase his expenses, he will have to pay each man 4s. less. How much does he intend to pay each man?

27. A greengrocer buys a certain number of melons for £1, 2s. 6d. He finds that three are bad, but by selling the rest at 3d. each more than they cost, he makes a profit of 3s. 6d. How many did he buy ?

28. AB is a straight line 4·7 cms. long. P is a point in AB such that $AP^2=2PB^2$. Find the length of AP correct to the nearest $\frac{1}{100}$th cm.

29. Answer Question 28 if P is a point on AB produced.

30. Find the speed of a motor-car if increasing the speed by 6 miles an hour means a saving of 12 minutes on a journey of 36 miles.

31. Two men walk towards one another starting from two towns 34 miles apart, and meet in 4 hours. If one takes 1 min. 40 sec. longer to walk a mile than the other, find their speeds.

32. I bought a number of pears for 10s. Later I passed a shop where the price of a pear was $\frac{1}{2}$d. less than I had paid, and calculated that I would have obtained 8 more pears for my money. How many pears did I buy ?

33. A man buys a number of water-jugs for £5, 10s. He breaks 4 of these, but is able to make £1, 10s. profit by selling the rest at 1s. each more than he paid. How much did he pay for each jug ?

34. £5 is divided equally between a number of boys. If there had been 4 boys fewer, each would have received 10d. more. How many boys were there ?

35. Two men walk from London to Barnet, a distance of 12 miles. The first walks at a uniform rate. The second, who starts half an hour later, rests for half an hour on the way, but while he is walking travels 1 mile an hour faster than the other. If the two arrive at Barnet together, how long did the first man take ?

36. Two cyclists ride from London to Eastbourne, a distance of 72 miles. One travels 35·2 inches further than the other every second, and takes 1 hour 12 minutes less for the journey. Find the speed of each.

37. A man can row 10 miles an hour in still water. If it takes him $1\frac{1}{4}$ hours to row 6 miles with the stream and then back again, find the speed of the stream.

38. AB is a straight line 3·8 inches long. Find a point X in AB such that $AX^2-AX \cdot XB=XB^2$. (Ans. to 2 decimal places.)

39. AB is a straight line 3·8 inches long. Find a point X on AB produced such that $AX^2 - AX \cdot BX = BX^2$. (Ans. to 2 decimal places.)

40. From London to St Albans is $20\frac{1}{2}$ miles. A starts from London, and B, walking $\frac{1}{2}$ m.p.h. faster than A, sets out from St Albans 20 minutes later. If the two meet 10 miles from London, find their speeds.

41. A company at an inn had £12 to pay, but before the bill was settled three of them left, so that those that remained had 4s. each more to pay. Of how many persons did the company consist?

42. A man bought a number of pictures for four guineas. He kept four pictures, and sold the rest at 1s. each more than he paid, thus gaining 6s. on his outlay. How many pictures did he buy?

43. A man buys a certain number of pencils for 2s. 6d. and sells them at 4 for 3d., thus gaining as much as he paid for 30 pencils. How much did he pay for each pencil?

44. What is the price of a fountain pen, if the number that could be obtained for £1 would be reduced by 2 if the price were to be increased by 6d.?

45. I bought a certain number of ties for a guinea. Had I bought a cheaper kind of tie at 3d. each less, I should have obtained two more ties for the money. Find the price I paid for a tie.

46. If 8 fewer handkerchiefs can be bought for 15s. when the price is raised 9d. a dozen, what is the price of a handkerchief?

47. A boy spends $7\frac{1}{2}$d. on toffee apples. Another boy buys smaller toffee apples, costing $\frac{1}{2}$d. each less, and gets an extra toffee apple for the same money. Find the price per apple in each case.

48. The breadth of a room is 3 ft. more than its height, and 5 ft. less than its length. If it measured 2 ft. more each way it would contain 1260 cubic feet more. Find the dimensions of the room.

Exercise 22c

1. There are two cubes whose edges differ by 3 inches and whose volumes differ by 657 cubic inches. Find the length of the side of the smaller cube.

2. A person pays 2d. more a cwt. for his coal than his neighbour, and when spending two guineas receives 3 cwt. more than

does his neighbour when spending £1, 12s. 6d. Find the prices paid for a cwt. of coal.

3. A number consists of two digits whose average is 5. If 13 be subtracted from three times the number, the result will be twice the square of the units digit. Find the number.

4. *A* and *B* working together can do a piece of work in three days. If they were working separately, *A* would take eight days longer than *B*. How long would *A* take alone ?

5. Two men had 50 sheep between them. The first sold all his sheep for £16, 13s. 4d., and the second sold his for £37, 10s. If each had sold his sheep at a price per sheep charged by the other one, they would have received equal sums of money. How many sheep had each ?

6. The length, breadth and height of a rectangular block are proportional to 9, 6 and 4. If the length were to be increased by 3 inches, the breadth reduced by 2 inches, and the height reduced by 1 inch, the volume would be reduced by 552 cubic inches. Find the original dimensions.

7. What is the price of herrings per dozen when two less in a shillingsworth raises the price by a penny a dozen ?

8. The tens digit of a number consisting of two digits is $1\frac{1}{3}$ times the units digit. If 30 be subtracted from the number, the result is equal to the sum of the digits multiplied by half the tens digit. Find the number.

9. The cost in pence of a dozen pears in one shop is three more than the number that can be bought for 2s. 6d. in another shop. By buying 100 pears at the first shop and selling them at the second shop, a profit of 4s. 2d. is made. Find the price of a pear at the first shop.

10. If a person whose stride is $2\frac{1}{5}$ feet were to take $\frac{1}{8}$ second longer over each step, his speed would be reduced by 1 mile an hour. At what rate is he walking ?

11. The expression ax^2+bx+c has the values $-1\frac{1}{2}$, 15, $40\frac{1}{2}$ when x has the values 2, 3, 4 respectively. Find the values of a, b and c. Find also the values of x which make the expression equal to zero.

12. A man invests his money at compound interest for two years at a certain rate per cent., and finds that he receives 5s. per cent. more than if he had invested it at simple interest. Find the rate per cent.

13. A bathing pool can be filled by two pipes together in 2 hours 24 minutes. The smaller alone takes 2 hours longer than the larger to fill the pool. How long does each pipe take alone ?

14. A man buys a bicycle and then sells it, making a profit which is five times as much per cent. as the number of pounds he paid. If he sells it for £1, 12s. 3d., how much did he pay for it ?

15. A man invests some money in a $1\frac{1}{2}$ per cent. stock. Had the price of the stock been increased by £3, his percentage annual return on his money would have been decreased by $1\frac{1}{2}$. Find the price of the stock.

16. A greengrocer finds that by selling a melon for 1s. $7\frac{1}{2}$d. his percentage of profit on cost price is twice as much as the number of pence the melon cost him. What did he pay for it ?

17. The hot pipe takes 3 minutes longer to fill a bath than the cold pipe. The two together take 6 minutes 40 seconds. How long does the cold pipe take ?

18. A number of two digits is such that twice the tens digit together with three times the units digit is equal to 20. If the square of the sum of the digits be decreased by the sum of the digits, the result is equal to the original number. Find the number.

19. A man buys a billiard table, and in reselling it finds that the percentage loss on the cost price is one quarter of the number of pounds he paid for it. If he sold the table for £51, what did he pay for it ?

20. Two men are cycling along a road in opposite directions; their speeds differ by 3 m.p.h., and they started from points 54 miles apart. After the moment of crossing, the slower takes $2\frac{1}{2}$ hours to reach the other one's starting-point ; find the speed of each.

21. How long will it take each of two pipes separately to fill a tank if one of them alone takes $9\frac{3}{8}$ minutes longer to fill the tank than the other, and 15 minutes longer than the two together ?

22. A man invests some money in a 6 per cent. stock. His friend invests later, when the price of the stock has increased by £5, and thus receives $\frac{1}{2}$ per cent. less on his investment. What was the original price of the stock ?

23. A dealer bought a certain number of eggs for two guineas, which he intended to sell at half a guinea profit. As he broke ten of them, he sold the remainder at a farthing each more than he had intended, thus making a profit of 16s. 4d. How many eggs did he buy, and at what price did he intend to sell each ?

24. A and B separately can do a piece of work in the same time ; C takes 4 days longer. If the three work together, the time taken is 9 days less than that of C. Find the time C takes.

25. A number consists of three digits. It is equal to the sum of the cubes of its digits, decreased by three times the product of its digits, and increased by twice the product of its middle digit and the difference of the squares of its tens and units digits. If the hundreds digit is one more, and the tens digit is three more than the units digit, find the number.

CHAPTER XXIII

Simultaneous Quadratic Equations

161. In Chapter XI it was shown that a set of n linear simultaneous equations containing n unknowns can, in general, be solved, and lead to a solution giving one value for each of the unknowns. We shall now deal with simultaneous equations which are not all linear.

Example 1. Solve $\quad 16x^2-9y^2=319 \quad . \quad . \quad . \quad . \quad (1)$

$\quad 4x-3y=11 \quad . \quad . \quad . \quad . \quad (2)$

Dividing corresponding sides of equations (1) and (2),

$$\frac{16x^2-9y^2}{4x-3y}=\frac{319}{11},$$

$$\text{i.e.} \quad \frac{(4x+3y)(4x-3y)}{4x-3y}=\frac{319}{11},$$

$$\therefore \quad 4x+3y=29 \quad . \quad . \quad . \quad . \quad (3)$$

(2)+(3) $\quad 8x=40,$

$$\therefore \quad x=5.$$

Substituting in equation (3), $\quad y=3.$

Thus the solution is $\quad x=5,\ y=3.$

Solving by Substitution

162. The method used in the above example is not always applicable. We shall now employ a method which is applicable in every case where one equation is of the second degree and the other linear. The method is to find the value of one unknown in terms of the other unknown from the linear equation, and to substitute this value for that unknown in the equation of higher degree.

Example 2. Solve $\quad x-3y=1 \quad . \quad . \quad . \quad . \quad (1)$

$\quad xy=4 \quad . \quad . \quad . \quad . \quad (2)$

Using equation (1) to substitute for x in terms of y,

$$x=1+3y \quad . \quad . \quad . \quad (3)$$

Substituting this value of x in equation (2), we have

$$(1+3y)y=4,$$

i.e. $$3y^2+y-4=0,$$

$$\therefore \quad (3y+4)(y-1)=0.$$

Hence, $$y=1, \text{ or } -\tfrac{4}{3}.$$

Substituting in equation (1), if $y=1$, $x=2$;

if $y=-\frac{4}{3}$, $x=-3$.

Thus the solution is $$x=2,\ y=1;$$
$$x=-3,\ y=-\tfrac{4}{3}.$$

Example 3. Solve $$3x^2+2xy-3y^2-3x+1=0 \quad . \quad . \quad . \quad (1)$$
$$2x+3y=1 \quad . \quad . \quad . \quad (2)$$

Using equation (2) to substitute for x in terms of y,

$$2x=1-3y,$$

$$\therefore \quad x=\frac{1-3y}{2} \quad . \quad . \quad . \quad . \quad (3)$$

Substituting this value of x in equation (1), we have

$$3\left(\frac{1-3y}{2}\right)^2+2\left(\frac{1-3y}{2}\right)y-3y^2-3\left(\frac{1-3y}{2}\right)+1=0,$$

i.e. $$\frac{3(1-6y+9y^2)}{4}+(1-3y)y-3y^2-3\left(\frac{1-3y}{2}\right)+1=0,$$

i.e. $$\frac{3-18y+27y^2}{4}+y-3y^2-\frac{3+9y}{2}+1=0.$$

Multiplying throughout by 4,

$$3-18y+27y^2+4y-12y^2-12y^2-6+18y+4=0,$$

i.e. $$3y^2+4y+1=0,$$

$$\therefore \quad (y+1)(3y+1)=0.$$

Hence, $$y=-1, \text{ or } -\tfrac{1}{3}.$$

Substituting in equation (3), if $y=-1$, $x=2$;

if $y=-\frac{1}{3}$, $x=1$.

Thus the solution is $$x=1,\ y=-\tfrac{1}{3};$$
$$x=2,\ y=-1.$$

Note (1).—In giving the solution, **the values of x and y must be arranged in their corresponding pairs.** The student should notice that the values $x=1$, $y=-1$ and the values $x=2$, $y=-\frac{1}{3}$ satisfy *neither* of the equations.

Note (2).—It will sometimes involve less work to use the linear equation to substitute for y in terms of x, for example when y has a smaller coefficient than x in the linear equation.

The questions in Exercise 23a, page 336, should now be attempted.

163. Example 4. Solve $\frac{1}{x^2}+\frac{1}{y^2}+\frac{5}{x}=5$ . . . . (1)

$$\frac{3}{x}+\frac{1}{y}=3 \quad . \quad . \quad . \quad . \quad (2)$$

Note.—We do not clear of fractions, but treat these equations as a quadratic and a linear equation in the unknowns $\frac{1}{x}$ and $\frac{1}{y}$.

As the coefficient of $\frac{1}{y}$ in equation (2) is unity, we shall substitute for $\frac{1}{y}$ in terms of $\frac{1}{x}$.

$$\frac{1}{y}=3-\frac{3}{x} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

Substituting this value of $\frac{1}{y}$ in equation (1),

$$\frac{1}{x^2}+\left(3-\frac{3}{x}\right)^2+\frac{5}{x}=5,$$

$$\text{i.e.} \quad \frac{1}{x^2}+9-\frac{18}{x}+\frac{9}{x^2}+\frac{5}{x}=5,$$

$$\text{i.e.} \quad \frac{10}{x^2}-\frac{13}{x}+4=0,$$

$$\therefore \quad \left(\frac{2}{x}-1\right)\left(\frac{5}{x}-4\right)=0.$$

Hence, $$\frac{1}{x}=\frac{1}{2}, \text{ or } \frac{4}{5}.$$

Substituting in equation (3), if $\frac{1}{x}=\frac{1}{2}, \frac{1}{y}=\frac{3}{2}$;

if $\frac{1}{x}=\frac{4}{5}, \frac{1}{y}=\frac{3}{5}$.

Hence the solution is $x=2, y=\frac{2}{3}$;

$x=\frac{5}{4}, y=\frac{5}{3}$.

164. It is sometimes possible, by division, to reduce an example containing an equation of higher degree than the second to one of the preceding types.

Example 5. Solve

$$a^3-b^3=152 \quad . \quad . \quad . \quad (1)$$

$$a-b=8 \quad . \quad . \quad . \quad (2)$$

Dividing corresponding sides of the equations (1) and (2),

$$\frac{a^3-b^3}{a-b}=\frac{152}{8},$$

i.e. $$\frac{(a-b)(a^2+ab+b^2)}{a-b}=\frac{152}{8},$$

$$\therefore \quad a^2+ab+b^2=19 \quad . \quad . \quad . \quad (3)$$

Using equation (2) to substitute for b in terms of a,

$$b=a-8, \quad . \quad . \quad . \quad (4)$$

Substituting this value of b in equation (3), we have

$$a^2+a(a-8)+(a-8)^2=19,$$

i.e. $$a^2+a^2-8a+a^2-16a+64=19,$$

i.e. $$3a^2-24a+45=0.$$

Dividing throughout by 3,

$$a^2-8a+15=0,$$

$$\therefore \quad (a-3)(a-5)=0.$$

Hence, $a=3$, or 5.

Substituting these values of a in equation (4),

$b=-5$, or -3.

Hence the solution is $a=3, b=-5$;

$a=5, b=-3$.

The questions in Exercise 23b, page 337, should now be attempted.

Homogeneous Equations

165. An expression is said to be **homogeneous** when every term in it is of the same degree.

Thus, $3x^3-4x^2y+y^3$ is a homogeneous expression of the third degree in x and y.

When the terms involving the unknowns are homogeneous in the second degree, none of the previous methods are applicable. The solution can be obtained, however, as in the following example:

Example 6. Solve

$$12x^2-28xy+27y^2=64 \quad . \quad . \quad . \quad . \quad (1)$$

$$16x^2-41xy+36y^2=78 \quad . \quad . \quad . \quad . \quad (2)$$

Dividing corresponding sides of equations (1) and (2),

$$\frac{12x^2-28xy+27y^2}{16x^2-41xy+36y^2}=\frac{64}{78}=\frac{32}{39},$$

$$\therefore \quad 39(12x^2-28xy+27y^2)=32(16x^2-41xy+36y^2),$$

i.e. $\quad 468x^2-1092xy+1053y^2=512x^2-1312xy+1152y^2,$

whence, $\quad 44x^2-220xy+99y^2=0.$

Dividing throughout by 11,

$$4x^2-20xy+9y^2=0,$$

$$\therefore \quad (2x-y)(2x-9y)=0.$$

Hence, $\quad 2x=y$, or $2x=9y$,

i.e. $\quad y=2x$, or $y=\dfrac{2x}{9}$.

When $y=2x$. Substituting this value of y in equation (1),

$$12x^2-28x(2x)+27(2x)^2=64,$$

i.e. $\quad 12x^2-56x^2+108x^2=64,$

i.e. $\quad 64x^2=64,$

$$\therefore \quad x^2=1, \qquad \therefore \quad x=\pm 1.$$

But $\quad y=2x, \qquad \therefore \quad y=\pm 2.$

When $y=\frac{2x}{9}$. Substituting this value of y in equation (1),

$$12x^2-28x\left(\frac{2x}{9}\right)+27\left(\frac{2x}{9}\right)^2=64,$$

$$\text{i.e.}\quad 12x^2-\frac{56x^2}{9}+\frac{4x^2}{3}=64.$$

Multiplying throughout by 9,

$$108x^2-56x^2+12x^2=576,$$

$$\text{i.e.}\qquad 64x^2=576,$$

$$\therefore\quad x^2=9,\qquad \therefore\quad x=\pm 3.$$

But $$y=\frac{2x}{9},\qquad \therefore\quad y=\pm\frac{2}{3}.$$

The solution is :

$x=1,\quad y=2;\qquad x=3,\quad y=\frac{2}{3};$

$x=-1,\quad y=-2;\qquad x=-3,\quad y=-\frac{2}{3}.$

The questions in Exercise 23c, page 338, should now be attempted.

Problems

166. We shall now consider some problems the solution of which depends on the methods given in this chapter.

Example 7. Find the dimensions of a rectangle whose area is 21 square inches and whose perimeter is 20 inches.

Denote the length and breadth of the rectangle by x and y inches respectively.

The area is xy sq. inches.

Hence, $$xy=21 \quad . \quad . \quad . \quad . \quad (1)$$

The perimeter is $(2x+2y)$ inches.

Hence, $$2x+2y=20,$$

$$\therefore\quad x+y=10 \quad . \quad . \quad . \quad . \quad (2)$$

From equation (2), $$y=10-x \quad . \quad . \quad . \quad . \quad (3)$$

Substituting this value of y in equation (1),

$$x(10-x)=21,$$

whence, $$x^2-10x+21=0,$$

$$\therefore \quad (x-3)(x-7)=0.$$

Hence, $$x=3, \text{ or } 7.$$

Substituting in equation (3), $y=7$, or 3.

Thus the dimensions of the rectangle are 3 inches and 7 inches.

Example 8. The interest on a certain sum of money for one year is £28. If the rate is reduced by $\frac{1}{2}\%$ and the sum of money increased by £100, the year's interest is still £28. Find the sum of money and the rate per cent.

Denote the sum of money by £x, and the rate by $y\%$.

Then, $$\frac{xy}{100}=28, \quad \therefore \ xy=2800 \quad . \quad . \quad (1)$$

Also, $$\frac{(x+100)(y-\frac{1}{2})}{100}=28,$$

$$\therefore \quad (x+100)(y-\tfrac{1}{2})=2800,$$

i.e. $$xy+100y-\tfrac{1}{2}x-50=2800 \quad . \quad . \quad . \quad . \quad (2)$$

It will be seen that neither (1) nor (2) is a linear equation.

A linear equation can be formed, however, by subtraction.

Thus, $$100y-\tfrac{1}{2}x-50=0,$$

whence, $$x=200y-100 \quad . \quad . \quad . \quad (3)$$

Substituting this value of x in equation (1),

$$(200y-100)y=2800,$$

i.e. $$200y^2-100y-2800=0,$$

$$\therefore \quad 2y^2-y-28=0,$$

$$\therefore \quad (y-4)(2y+7)=0.$$

Hence, $$y=4, \text{ or } -\tfrac{7}{2}.$$

Rejecting the negative root, which is inadmissible, and substituting the root $y=4$ in equation (3),

$$x=700.$$

Hence the sum of money is £700, and the rate is 4%.

Example 9. A vehicle covers a distance of 30 miles in half an hour less than a second vehicle. Had the speed of the first been increased by 3 m.p.h., it would have taken one hour less than the second vehicle. Find their speeds.

Denote their speeds by x and y m.p.h.

Then, $$\frac{30}{x}=\frac{30}{y}-\frac{1}{2} \quad . \quad . \quad . \quad . \quad (1)$$

If the speed of the first is increased to $(x+3)$ m.p.h., the difference in the times is one hour.

Hence, $$\frac{30}{x+3}=\frac{30}{y}-1 \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating y by subtraction,

$$(1)-(2),\ \frac{30}{x}-\frac{30}{x+3}=\frac{1}{2}.$$

Multiplying throughout by $2x(x+3)$,

$$60(x+3)-60x=x(x+3),$$

whence, $$x^2+3x-180=0,$$

$$\therefore\quad (x-12)(x+15)=0.$$

Hence, $$x=12, \text{ or } -15.$$

Rejecting the negative root, which is inadmissible, and substituting the root $x=12$ in equation (1),

$$y=10.$$

Thus their speeds are 12 and 10 m.p.h. respectively.

The questions in Exercise 23d, page 339, should now be attempted.

Exercise 23a

Solve the simultaneous equations:

1. $4x^2-y^2=57,$
$2x-y=19.$

2. $25y^2-x^2=99,$
$5y+x=9.$

3. $3x-7y=-2,$
$9x^2-49y^2=32.$

4. $x^2-4y^2=10,$
$x+2y=2.$

5. $6x^2+7xy+2y^2+8=0,$
$2x+y-4=0.$

6. $x(3x-8y)+48+4y^2=0,$
$3(x-4)=2y.$

7. $x^2-y^2=7=x+y.$

8. $4x^2-9y^2=3y-2x=-5.$

9. $2x^2-3xy-9y^2=-19,$
$x-3y=1.$

10. $5x^2+\frac{1}{4}=3xy,$
$5x-3y+\frac{1}{2}=0.$

11. $x^2+y^2=13,$
$x-y=1.$

12. $x-y=3,$
$xy=28.$

13. $xy=1,$
$x-2y=1.$

14. $x-3y=12,$
$x^2-3y^2=-18.$

15. $xy+10=0,$
$3x+y=-1.$

16. $x^2-y=5,$
$y-3x=-7.$

17. $x+y-7=0=xy-6.$

18. $2x^2+y^2=9,$
$x+2y=4.$

19. $5x^2+3xy+y^2=15,$
$7x+2y=12.$

20. $x^2-y^2=1+xy,$
$3x+1=-y+4.$

Exercise 23b

Solve the simultaneous equations :

1. $\frac{1}{x^2}-\frac{1}{y^2}=3,$
$\frac{1}{x}-\frac{1}{y}=1.$

2. $\frac{4}{x^2}-\frac{9}{y^2}=-3,$
$\frac{2}{x}+\frac{3}{y}=3.$

3. $\frac{9}{x^2}-\frac{1}{y^2}=-\frac{3}{4},$
$\frac{3}{x}-\frac{1}{y}=\frac{3}{2}.$

4. $\frac{1}{x}+\frac{1}{y}=2,$
$\frac{1}{x^2}-\frac{1}{y^2}=0.$

5. $\frac{1}{x^2}+\frac{1}{y^2}=1,$
$\frac{1}{x}-\frac{1}{y}=\frac{1}{5}.$

6. $\frac{1}{x}+\frac{1}{y}=5,$
$\frac{1}{x^2}+\frac{1}{y^2}=13.$

7. $\frac{3}{x^2}+\frac{1}{y^2}=16,$
$\frac{2}{x}-\frac{1}{y}=-6.$

8. $\frac{2}{x^2}-\frac{5}{y^2}=13,$
$\frac{3}{x}-\frac{4}{y}=5.$

9. $\frac{2}{x^2}+\frac{7}{xy}+\frac{3}{y^2}=22\frac{2}{9},$
$\frac{1}{x}+\frac{3}{y}=-6\frac{2}{3}.$

10. $\frac{6}{x^2}-\frac{13}{xy}-\frac{5}{y^2}=-12,$
$\frac{2}{x}-\frac{5}{y}=3.$

11. $\frac{1}{x^2}+\frac{3}{x}+\frac{1}{y^2}=-1,$
$\frac{1}{x}+1=\frac{1}{y}.$

12. $\frac{2}{x^2}+\frac{1}{y^2}+\frac{5}{x}=7,$
$\frac{2}{x}-\frac{1}{y}=3.$

13. $x^3+y^3=133,$
$x+y=7.$

14. $x^3+8y^3=56,$
$x+2y=2.$

15. $27x^3-8y^3=35,$
$3x-2y=5.$

16. $15x-3y=-3,$
$250x^3-2y^3=-38.$

17. $\frac{1}{x^3}+\frac{1}{y^3}=\frac{7}{8},$
$\frac{1}{x}+\frac{1}{y}=\frac{1}{2}.$

18. $\frac{1}{x}-\frac{1}{2y}=-\frac{1}{2},$
$\frac{4}{x^3}-\frac{1}{2y^3}=-\frac{37}{2}.$

19. $x+y=5,$
$\frac{1}{x}+\frac{1}{y}=\frac{5}{4}.$

20. $\frac{1}{x}+\frac{1}{y}=\frac{10}{3},$
$x+y=\frac{10}{3}.$

21. $\frac{1}{x}-\frac{1}{y}=1,$
$x-y=-\frac{1}{2}.$

22. $2x+3y=13,$
$\frac{5}{x}+\frac{4}{y}=3\frac{5}{6}.$

23. $2y^2-8x^2+3y=5(x-1),$
$2x-y=1.$

24. $3y+x+2=0,$
$\frac{2x^2+2y+3y^2}{5x(1-3x)+7}=\frac{1}{3}.$

25. $\frac{2(x-1)}{y-1}+\frac{y+1}{2x+6}=\frac{17}{15},$
$-2x+y=3.$

26. $\frac{5(y+1)}{x}-\frac{4-6x}{5y-2}=10,$
$x-1=\frac{5}{2}y.$

27. $\frac{1-2y}{x+1}-\frac{2x+1}{2y-3}=2\frac{1}{4},$
$x+2y+1=0.$

28. $\frac{y-1}{2x-3}-\frac{3(7-4x)}{4y-5}=\frac{7}{2},$
$3x-y-1=0.$

29. $\frac{8y+9}{5x-1}=\frac{2y}{2x-1},$
$3-x+2y=0.$

30. $\frac{5x+4y}{5x+6y}+\frac{14y+11x}{7x+6y}=\frac{7}{2},$
$x+2y=-1.$

31. $\frac{1}{7x-3y}+\frac{3}{13x-3y}=\frac{2}{3y-x},$
$5x-3y=2.$

32. $\frac{5x-y}{y-4x}-\frac{x+2y}{7y-12x}=y-1-2x=0.$

Exercise 23c

Solve :

1. $6x^2-4xy+y^2=27,$
$3x^2-2xy+2y^2=27.$

2. $4x^2-2xy-y^2=44,$
$4x^2-6xy+3y^2=28.$

3. $x^2+3xy+y^2=11,$
$x^2+xy+y^2=7.$

4. $x^2+xy+2y^2=x^2+2xy+y^2=4.$

5. $x^2+6xy+4y^2=44,$
$x^2+2xy+4y^2=28.$

6. $12x^2-52xy+67y^2=64,$
$16x^2-73xy+93y^2=78.$

7. $3x^2-3xy+2y^2=72,$
$4x^2-5xy+2y^2=99.$

8. $3x^2-9xy+8y^2=18,$
$16x^2-52xy+44y^2=99.$

9. $7x^2-3xy=-\frac{1}{2}$, $3y^2-6xy-2x^2=1\frac{3}{4}$.

10. $4x^2-5xy+y^2=2$, $5x^2-xy=\frac{7}{9}$.

11. $2x^2-3xy+y^2=6$, $13x^2-22xy+10y^2=36\frac{1}{4}$.

12. $5x^2+6xy+2y^2=38\frac{1}{4}$, $22x^2+29xy+7y^2=-153$.

13. $2x^2-3xy+y^2=30$, $x^2-xy+y^2=21$.

14. $x^2+xy+y^2=3$, $x^2-xy+y^2=1$.

15. $7x^2+3xy-y^2=18$, $3x^2-xy+y^2=10$.

16. $x^2+2xy+3y^2=1$, $3x^2-4xy+5y^2=3$.

17. $8x^2-3xy+3y^2=42$, $5x^2+xy=21$.

18. $4x^2+11xy-2y^2=4$, $12x^2-9xy+3y^2=6$.

19. $x(2x-3y)-11=-2y^2$, $2(1-2xy+x^2)=x^2-y^2$.

20. $2x-5y=\dfrac{30}{x-3y}$,

21. $7x-3y-\dfrac{7}{5x-2y}=0$, $\qquad x-5y=\dfrac{7(3-y^2)}{x}$.

$35x^2-29xy+6y^2-24\frac{1}{2}=(2y-5x)(11x-7y)$.

Solve the following equations for x and y :

22. $x^2-y^2=a^2$, $x+y=ab$.

23. $xy=7m^2$, $6y-x=19m$.

24. $x^3+y^3-7a^3=x+y-a=0$.

25. $x^2+2xy+2y^2=\dfrac{5}{b^2}$, $y^2+xy=\dfrac{2}{b^2}$.

Exercise 23d

1. A number consists of two digits such that the sum of the digits is 8, and the difference of the squares of the digits is 16. What is the number?
2. The product of two numbers is 48. If the smaller number is subtracted from three-fourths of the greater, the result is 5. Find the numbers.
3. Find two numbers such that their product multiplied by their sum is 126, while their product multiplied by their difference is 70.
4. Two squares differ in area by 11 square feet. Find their sides if the perimeters differ by 4 ft.

5. Find two numbers such that their sum multiplied by the sum of their squares is 803, and their difference multiplied by the difference of their squares is 275.

6. A floor of a room has a carpet whose area is 99 sq. ft., and a border 18 inches wide, which has an area of 69 sq. ft., surrounds the carpet. What are the dimensions of the carpet ?

7. The product of two numbers is 60, and the product of their sum and difference is 209 ; find them.

8. A and B buy apples, B paying a $\frac{1}{4}$d. an apple more than A. A spends 1s. $10\frac{1}{2}$d., and B, who spends 2s. 4d., receives one apple more than A. How many apples did A buy ?

9. A tank, 4 feet deep, contains 220 cubic feet of water. Had it been 1 foot shorter but $\frac{1}{2}$ ft. wider, its capacity would have been unaltered. Find its length and breadth.

10. The sum of the volumes of two cubes is 152 c. ft., and the area of a face of each added together is 19 sq. ft. more than the area of the rectangle formed by an edge of each cube. What are the dimensions of these cubes ?

11. Two boys each spend 8d. on marbles, one getting 8 marbles more than the other. Had they bought 30 marbles each at the same prices, the first boy would have spent a penny less than the other boy. How many marbles did he get for his 8d. ?

12. Two gardens each have an area of 260 sq. yards. One is 4 feet narrower, but 12 feet longer, than the other. Find the width of the narrower garden.

13. In a right-angled triangle the area of the square on the hypotenuse is 65 sq. inches, and the difference in area of the squares on the sides containing the right angle is 33 sq. inches. What are the lengths of the sides containing the right angle ?

14. Two tobacconists each spend ten guineas in buying boxes of cigarettes. One of them pays a penny less per box, and so obtains 12 boxes more for his money. How many boxes does he obtain ?

15. A boy spent all his money on oranges : if he had obtained 5 oranges more, each would have cost $\frac{1}{2}$d. less, if he had obtained 3 fewer, each would have cost $\frac{1}{2}$d. more. How much did he spend ?

16. I spend a certain sum of money on oranges. If they had been 1d. per dozen dearer, I would have received 6 fewer for the same outlay. If, however, they had been 2d. per dozen cheaper, I would have received 6 more for $\frac{5}{6}$ of my outlay. What was the price per dozen, and what was my outlay ?

17. If increasing the speed by 10 m.p.h. means a saving of 25 minutes on a journey, and decreasing the speed by 10 m.p.h. means a loss of $37\frac{1}{2}$ minutes on the same journey, find the length of the journey and the speed.

18. A field has an area of 2451 sq. yds. An adjacent field, 2 yds. longer and 5 yds. wider, has an area of 2832 sq. yds. Find the dimensions of the latter field.

19. In a two-mile race one cyclist beat another by 4 minutes. Had the slower cyclist ridden at 2 m.p.h. faster, he would still have lost by 2 minutes. Find the time taken by the winner.

20. The interest on a sum of money for one year is £22, 10s. If the rate of interest were less by $\frac{1}{2}$ per cent., it would be necessary to invest £50 more to produce the same amount of interest. Find the sum invested at first.

21. A man bought a certain number of pears for a certain number of pence. Had he obtained five more for the same money, each would have cost $\frac{3}{4}$d. less; but had he obtained 5 fewer, each would have cost $1\frac{1}{2}$d. more. Find the number of pence.

22. A number consists of two digits such that the sum of the cubes of the digits is equal to 49 times the sum of the digits. Further, the number exceeds twice the product of its digits by the digit in the units place. Find the number.

23. A garden has an area of 3000 sq. ft. Along the two sides and at one end there is a path 4 ft. wide, the total area of the path being 728 sq. ft. Find the length of the garden.

24. The interest on a sum of money for one year is £27, 12s. 6d. If the sum were to be reduced by £25, but the rate of interest increased by $\frac{1}{4}$ per cent., the year's interest would be increased by 10s. Find the sum of money and the original rate.

25. The sum of two numbers taken from their product is 19, while the sum of the numbers added to the sum of their squares is 144. Find the numbers.

26. A train travels a certain distance at a uniform speed. Had the speed been 2 m.p.h. greater, the time taken would have been reduced by 5 minutes; had the speed been 5 m.p.h. slower, the time taken would have been increased by 15 minutes. Find the distance and the speed at which the train travelled.

27. Two stations A and B are 210 miles apart. A fast train leaves A for B; at the same time a train leaves B for A,

travelling 15 m.p.h. slower. A passenger in the fast train finds that he reaches B $1\frac{1}{2}$ hours after passing the slow train. What are the speeds of the two trains?

28. A plank of wood has 6 inches waste after being cut into a number of pieces of equal length. If each piece had been 1 inch shorter, a waste of 2 inches would have resulted, but the number of pieces would have increased by 1. If the original plank had been $1\frac{1}{2}$ times as long, and the pieces had been 3 inches longer than at first, two more pieces could have been cut without waste. How many pieces were there at first, and how long was each?

29. The hot and cold pipes together fill a bath in $3\frac{3}{4}$ minutes. The sum of the times each takes separately to fill the bath is 16 minutes. How long does each take alone?

30. The hypotenuse of a right-angled triangle is less than the sum of the other two sides by 4 inches, while the area of the triangle is 30 sq. inches. Find the length of the sides.

31. A train travels from A to B at 3 m.p.h. below its usual rate, thus arriving 9 minutes late at B. From B to C, a distance 14 miles less than AB, the driver increases the speed to 5 m.p.h. above the normal, and thus arrives at C one minute before schedule. Find the distance AC, and the normal speed.

32. A motor-car takes one hour longer for a certain distance if its speed is reduced by 5 m.p.h. When the speed is increased by 6 m.p.h., and the distance doubled, the time is 50 minutes more than $1\frac{1}{2}$ times that for the original distance. Find the speed of the car and the distance.

33. Find two quantities such that their sum, their product, and the difference of their squares are all equal.

34. The front wheels of a carriage are 21 inches less in circumference than the back, and make 36 more revolutions than the back wheels in a certain distance. If the circumference of the front wheels were 2 inches more, and that of the back wheels 6 inches less, then the front wheels would make 24 revolutions more than the back if the distance were increased by 8 yards. Find the circumference of the wheels, and the distance.

CHAPTER XXIV

Harder Graphs

Solving Quadratic Equations

167. In Chapter XIII it was shown how various statistical problems and also simultaneous linear equations may be solved graphically. We shall now deal with the graphs of quadratic functions.

Example 1. Draw the graph of the function $2x^2-x-6$, for the range $x=-3$ to $x=+3$, and thus solve the equation $2x^2-x-6=0$. Denote the function of x by y.

Then $y=2x^2-x-6$.

The table of values is:

$x=$	-3	-2	-1	0	1	2	3
$y=2x^2-x-6=$	15	4	-3	-6	-5	0	9

Fig. 34 shows the graph drawn to a scale of 1 inch=2 for x, and 1 inch=5 for y.

The values of x and y for any point on the graph are connected by the relation $y=2x^2-x-6$. Hence to find a value of x for which $2x^2-x-6$ has any required magnitude, we have merely to find a point on the graph for which the value of y has that magnitude, and then read off the corresponding value of x. Thus to solve the equation of $2x^2-x-6=0$, we have to find a point on the graph for which y has the magnitude 0, and then read off the corresponding value of x.

Now y has the magnitude 0 for all points on the line $X'OX$. Hence **a solution of the equation is obtained by the value of x for any point on the graph which lies on the line $X'OX$.** It will be seen that there are two such points, namely P and Q, the values of x at which are $-1\frac{1}{2}$ and 2 respectively. Hence the solution of the equation is $x=-1\frac{1}{2}$ or 2.

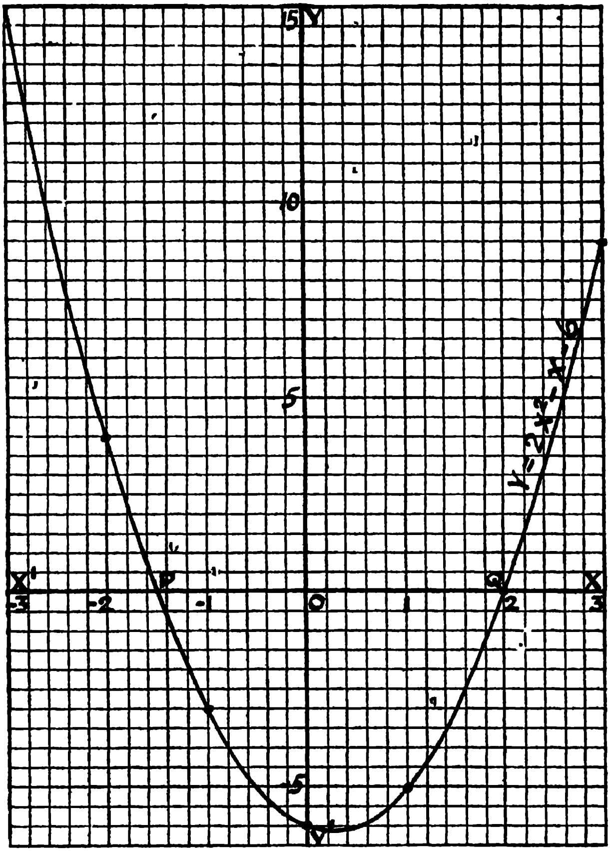

FIG. 34

Maximum Value of a Quadratic Function

168. Example 2. Plot the graph of the function $f(x)=1+12x-4x^2$, for the range $x=-1$ to 4, and thus find the maximum value of the function $1+12x-4x^2$, and the value of x which makes this function a maximum.

Denote the function $f(x)$ by y. Then $y=1+12x-4x^2$. The table of values is:

$x=$	-1	0	1	2	3	4
$y=1+12x-4x^2=$	-15	1	9	9	1	-15

Fig. 35 shows these points plotted to a scale of 1 inch=2 for x, and 1 inch=10 for y, but no graph has been drawn.

It will be noticed that the curve will bend rapidly for the range $x=0$ to 3, particularly between the values $x=1$ and $x=2$. It is thus difficult to draw the graph with any accuracy unless further points in this range are plotted. We shall therefore plot for half values of x throughout the range $x=0$ to 3, and for quarter values for the range $x=1$ to 2. We thus require the additional table of values:

$x=$	$\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{1}{2}$	$1\frac{3}{4}$	$2\frac{1}{2}$
$y=1+12x-4x^2=$	6	9·75	10	9·75	6

Such **additional values should always be plotted for any range where rapid bending occurs.** Fig. 36 shows all the points plotted and the graph drawn.

The values of x and y at any point on the graph are connected by the relation $y=1+12x-4x^2$. Hence to find the value of x which makes the function $1+12x-4x^2$ assume its maximum value, we must find the value of x at the point on the graph for which y has its maximum value, that is to say, at the highest point of the graph. It is seen that the point P, where $x=1\frac{1}{2}$, $y=10$, is the highest point on the graph. Hence the maximum value of the function $1+12x-4x^2$ is 10, and it has this maximum value when $x=1\frac{1}{2}$.

Minimum Value of a Quadratic Function

169. Example 3. Plot the graph of $y=3x^2+6x-11$, for the range $x=-4$ to 2, and from the graph read off the minimum value of the function $3x^2+6x-11$, and also the range of values of x for which the function is negative.

If a table for integral values of x be drawn up and plotted, it will be seen that the graph will bend rapidly for the range $x=-3$ to 1. Hence the table of values will be calculated for half values of x for the range $x=-3$ to 1, and quarter values of x for the range $x=-2$ to 0. The values of y, where not integral, are given to two places of decimals for convenience in plotting.

$x=$	-4	-3	$-2\frac{1}{2}$	-2	$-1\frac{3}{4}$
$y=3x^2+6x-11=$	13	-2	$-7·25$	-11	$-12·31$
$x=$	$-1\frac{1}{2}$	$-1\frac{1}{4}$	-1	$-\frac{3}{4}$	$-\frac{1}{2}$
$y=3x^2+6x-11=$	$-13·25$	$-13·81$	-14	$-13·81$	$-13·25$
$x=$	$-\frac{1}{4}$	0	$\frac{1}{2}$	1	2
$y=3x^2+6x-11=$	$-12·31$	-11	$-7·25$	-2	13

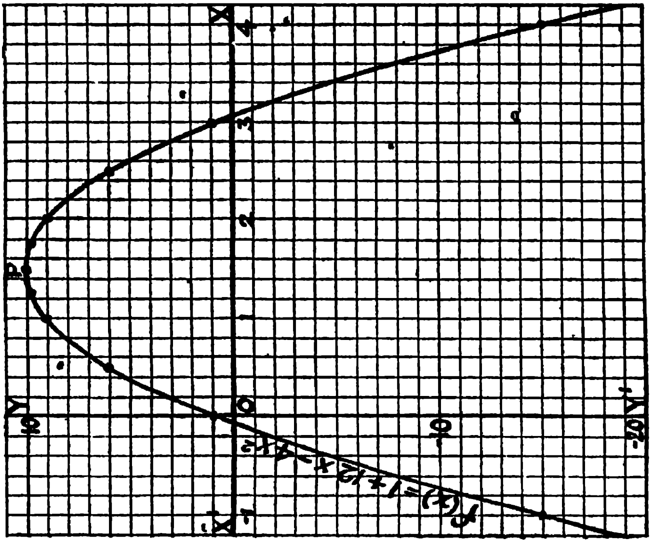

FIG. 36

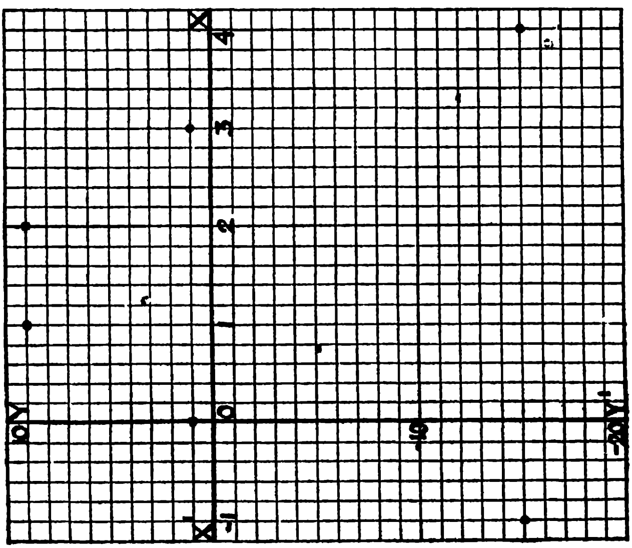

FIG. 35

Fig. 37 shows all these points plotted, to a scale of 1 inch=2 for x, and 1 inch=10 for y, and the graph drawn.

The minimum value of the function $3x^2+6x-11$ is the least value of y for any point on the graph, that is, the value of y at

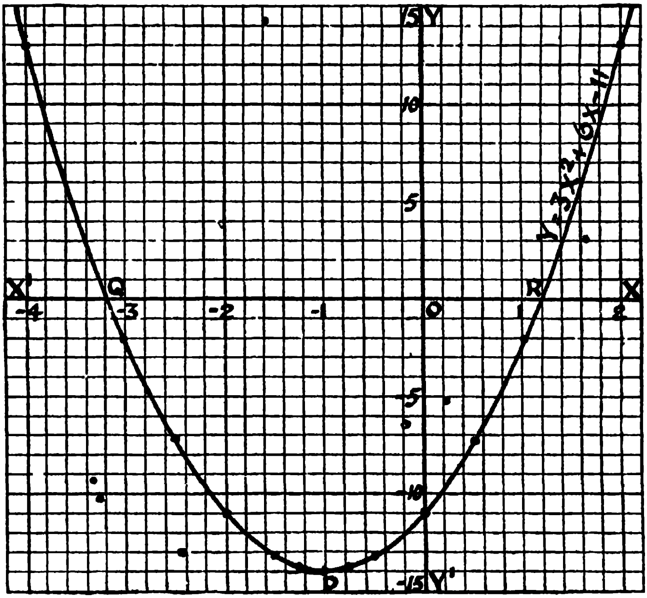

FIG. 37

the lowest point on the graph. It is seen that the lowest point on the graph is the point P, the value of y at which is -14. Thus the minimum value of the given function is -14.

Sign of a Quadratic Function

170. The function $3x^2+6x-11$ is negative for all points on the graph for which y is negative, that is, for all points on the graph which lie below the axis $X'OX$. It is seen that any value of x lying within the range QR gives a negative value for y, where the values of x at Q and R are $-3{\cdot}15$ and $1{\cdot}15$ approximately.

Thus the function $3x^2+6x-11$ is negative for all values of x lying within the range $-3{\cdot}15$ to $1{\cdot}15$.

Note.—It follows from the above that the function $3x^2+6x-11$ is positive for values of x outside the range $-3{\cdot}15$ to $1{\cdot}15$, that is, for any value of x less than $-3{\cdot}15$ or greater than $1{\cdot}15$, no matter how large numerically. These ranges are written $-\infty$ to $-3{\cdot}15$, and $1{\cdot}15$ to ∞, where ∞ is read **infinity.**

The questions in Exercise 24a, page 356, should now be attempted.

Graph of $y=x^2$ to Solve any Quadratic Equation

171. Example 4. Plot the graph of $y=x^2$, for the range $x=-3$ to 3. Use this graph to solve the equations (1) $x^2+x-2=0$; (2) $2x^2-2x-7=0$.

The table of values for the graph of $y=x^2$ is :

$x=$	-3	-2	$-1\frac{1}{2}$	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	0
$y=x^2=$	9	4	2·25	1	·56	·25	·06	0

$x=$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	2	3
$y=x^2=$	·06	·25	·56	1	2·25	4	9

The curved line on Fig. 38 shows these points plotted to a scale of 1 inch$=2$ for x, and 1 inch$=2$ for y, and the graph drawn.

(1) Now the equation $x^2+x-2=0$ may be put in the form $x^2=-x+2$. The co-ordinates of any point on the curved line already drawn are connected by the relation $y=x^2$. We shall now draw the graph of $y=-x+2$ **on the same axes and with the same scales.** The graph of $y=-x+2$ is a straight line, so that we need plot two points only. The table of values is :

$x=$	-3	2
$y=-x+2=$	5	0

These points are plotted and the straight line drawn on Fig. 38.

At a point where these two graphs intersect, that is, at a point which lies on both graphs, the co-ordinates of that point satisfy both the relation $y=x^2$ and the relation $y=-x+2$. Hence the co-ordinates of this point give a solution of the simultaneous equations $y=x^2$ and $y=-x+2$. Thus the value of x at such a

point of intersection satisfies the equation $x^2=-x+2$, that is, $x^2+x-2=0$. It is seen that the graphs cut at the two points

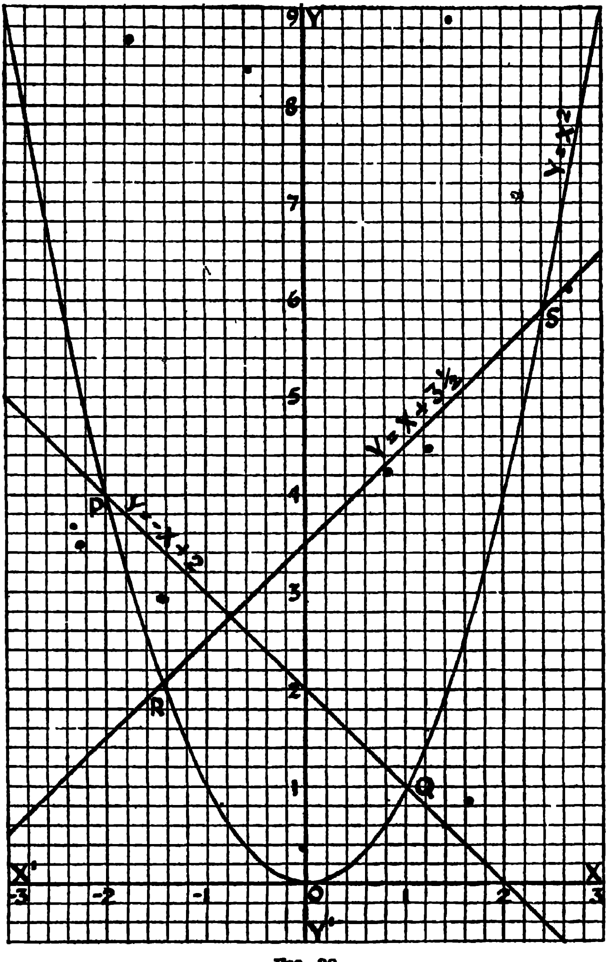

FIG. 88

P and *Q*, that is, at the points $x=-2$, $y=4$; and $x=1$, $y=1$ respectively.

For the purpose of emphasis, we restate the reasoning involved in obtaining a solution of equation (1).

The point P, $(-2, 4)$, lies both on the graph $y=x^2$ and on the graph $y=-x+2$. Hence the value $x=-2$ makes both x^2 and $-x+2$ assume the same value, namely 4. Thus the value $x=-2$ makes x^2 equal to $-x+2$. Therefore $x=-2$ is a root of the equation $x^2+x-2=0$. Similarly, a second root is $x=1$, corresponding to the x co-ordinate of the point Q.

Note.—The equation $x^2+x-2=0$ could have been solved by plotting the graph of $y=x^2+x-2$, and obtaining the points of intersection of the graph with the x-axis. The pupil will usually find, however, that **less work is involved in plotting the easier graph $y=x^2$, and the appropriate straight line,** as in this example.

(2) The equation $2x^2-2x-7=0$ may be put in the form $2x^2=2x+7$, or $x^2=x+3\frac{1}{2}$. To obtain a solution of this equation from the graph of $y=x^2$, we must now draw the graph of $y=x+3\frac{1}{2}$, on the same axes and with the same scales, and obtain the x co-ordinates of the points of intersection of the two graphs. The table of values is:

$x=$	-3	3
$y=x+3\frac{1}{2}=$	$\cdot5$	$6\cdot5$

These points are plotted and the straight line drawn on Fig. 38.

It is seen that this graph cuts the graph of $y=x^2$ at the points R and S, the x co-ordinates of which are $-1\cdot45$ and $2\cdot45$ respectively. Thus the solution of the equation $x^2=x+3\frac{1}{2}$, that is of the equation $2x^2-2x-7=0$, is $x=-1\cdot45$ or $2\cdot45$.

More Accurate Reading of the Roots

172. We may be required to obtain the solution of an equation to a greater degree of accuracy than is possible with the scales necessary to show the complete graphs on one diagram. In such a case, **the parts of the graphs near each point of intersection are enlarged separately** by using a larger scale. In the neighbourhood of the negative root $-1\cdot45$ above, for example, we have the following table of values for the two graphs, where x is made to change by 0·01 units:

$x=$	$-1\cdot46$	$-1\cdot45$	$-1\cdot44$	$-1\cdot43$
$x^2=$	$2\cdot132$	$2\cdot103$	$2\cdot074$	$2\cdot045$
$x+3\frac{1}{2}=$	$2\cdot04$			$2\cdot07$

Fig. 39 shows these points plotted, and the two graphs drawn, to a scale of 1 inch$=0\cdot01$ for x, and 1 inch$=0\cdot04$ for y.

It is seen that the two graphs intersect at the point R, the x co-ordinate of which is $-1\cdot4365$. Thus $x=-1\cdot4365$ is a more

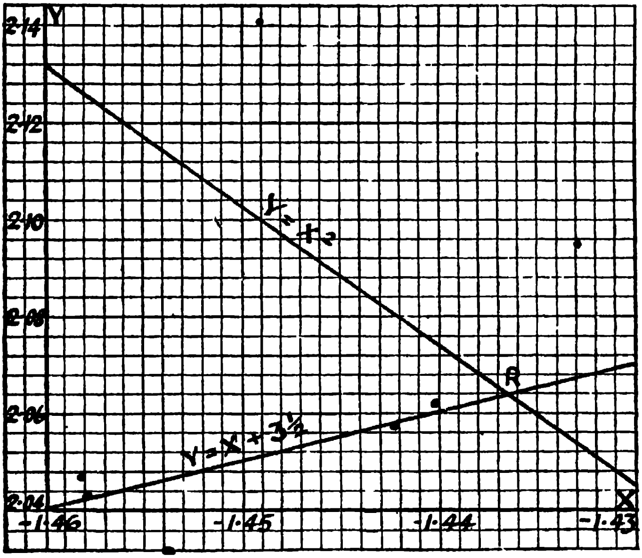

FIG. 39

accurate approximation to the root. The pupil may similarly obtain a closer approximation to the root $x=2\cdot45$ above, and then verify each of these closer approximations by solving the equation $2x^2-2x-7=0$ algebraically to four places of decimals.

Equal Roots and Imaginary Roots

173. Example 5. Plot the graph of $y=x^2$ for the range $x=-3$ to 3. Use this graph to solve the equations (1) $4x^2-12x+9=0$; (2) $x^2+x+1=0$. (3) Find also from the graph for what range of values of x the function $2x^2-x-10$ is negative.

The table of values for the graph of $y=x^2$ is that given in Example 4. The curved line in Fig. 40 shows these points plotted

and the graph drawn, to a scale of 1 inch=2 for x, and 1 inch=4 for y.

(1) The equation $4x^2-12x+9=0$ may be put in the form $4x^2=12x-9$, or $x^2=3x-2\frac{1}{4}$. To obtain a solution of this equation from the graph of $y=x^2$, we must now plot the graph of $y=3x-2\frac{1}{4}$. The table of values is :

$x=$	0	3
$y=3x-2\frac{1}{4}=$	$2\frac{1}{4}$	$6\frac{3}{4}$

The graph of this straight line is shown in Fig. 40.

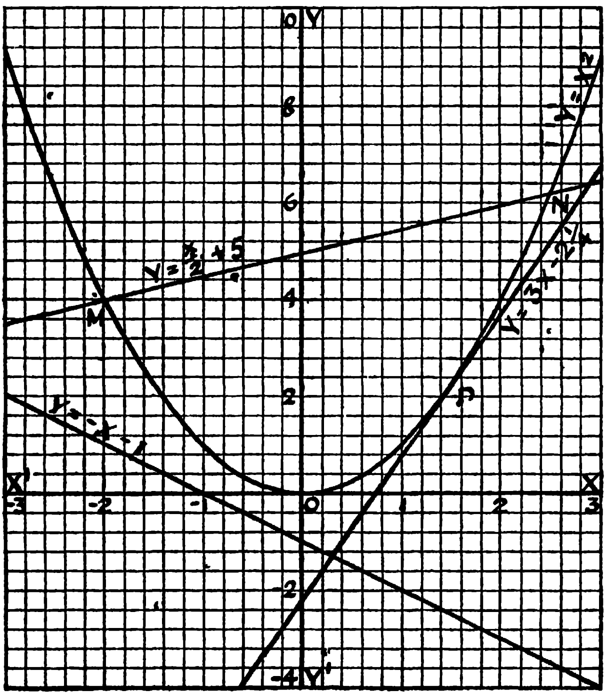

FIG. 40

It will be seen that the straight line is a **tangent** to the graph of $y=x^2$ at the point P whose x co-ordinate is $1\frac{1}{2}$. A tangent to a curve passes through two coincident points on the curve at the point of contact. Thus in our case the straight line meets

the curve at two coincident points at P. Hence the solution of the equation $x^2=3x-2\frac{1}{4}$, that is of the equation $4x^2-12x+9=0$, consists of two equal values of x, each being $1\frac{1}{2}$.

(2) The equation $x^2+x+1=0$ may be put in the form $x^2=-x-1$. We therefore draw a graph of $y=-x-1$, the table of values for which is:

$x=$	-3	1
$y=-x-1=$	2	-2

The graph of this straight line is shown in Fig. 40. The student will observe that there are no points common to this straight line and the graph of $y=x^2$. Hence there is no solution of the equation $x^2=-x-1$, that is of the equation $x^2+x+1=0$. More correctly, we ought to say that the solution of the equation $x^2+x+1=0$ is **imaginary**. (Compare § 155, Case 4.)

(3) $2x^2-x-10$ is negative for all values of x which make $2x^2$ less than $x+10$, that is which make x^2 less than $\frac{1}{2}x+5$. In Fig. 40 the graph of $y=x^2$ has been plotted. We now plot the graph of $y=\frac{1}{2}x+5$. The table of values is:

$x=$	-3	3
$y=\frac{1}{2}x+5=$	$3\frac{1}{2}$	$6\frac{1}{2}$

The graph of this straight line is shown in Fig. 40. Now x^2 is less than $\frac{1}{2}x+5$ for any value of x for which the ordinate to the curved graph is less than the ordinate to the straight line, that is for which the curved graph lies below the straight line. It is seen that the curved graph lies below the straight line for all values of x between those corresponding to the points M and N, that is between the values $x=-2$ and $x=2\frac{1}{2}$. Hence x^2 is less than $\frac{1}{2}x+5$, that is $2x^2-x-10$ is negative, for the range $x=-2$ to $2\frac{1}{2}$.

Graph of any Quadratic Function to Solve any Quadratic Equation

174. Example 6. Plot the graph of $y=5-3x-x^2$ for the range $x=-5$ to 3, and by use of the graph, (1) find the maximum value of the function $5-3x-x^2$, and (2) solve the equation $4x^2-4x-23=0$.

The table of values for $y=5-3x-x^2$ is:

$x=$	-5	-4	-3	$-2\frac{1}{2}$	-2	$-1\frac{3}{4}$	$-1\frac{1}{2}$	$-1\frac{1}{4}$	-1	$-\frac{1}{2}$	0	1	2	3
$y=5-3x-x^2=$	-5	1	5	$6{\cdot}25$	7	$7{\cdot}19$	$7{\cdot}25$	$7{\cdot}19$	7	$6{\cdot}25$	5	1	-5	-13

The graph is shown in Fig. 41.

(1) The highest point on this graph is the point P, where $y=7\cdot25$. Thus the maximum value of the function $5-3x-x^2$ is $7\cdot25$.

(2) The equation $4x^2-4x-23=0$ may be rewritten $4x^2=4x+23$, or $x^2=x+5\frac{3}{4}$.

$$\therefore \quad 5-3x-x^2=5-3x-(x+5\tfrac{3}{4}).$$

$$\text{i.e.} \quad 5-3x-x^2=-4x-\tfrac{3}{4}.$$

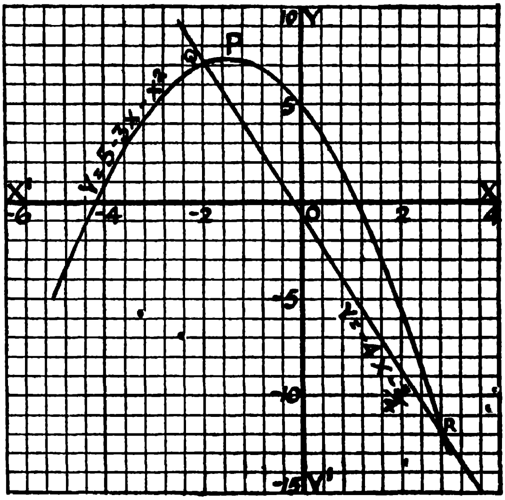

FIG. 41

Hence, as we already have the graph of $y=5-3x-x^2$, we have only to draw the graph of $y=-4x-\frac{3}{4}$, and obtain the co-ordinates of x at their points of intersection. The table of values for the straight line graph, $y=-4x-\frac{3}{4}$, is :

$x=$	-2	3
$y=-4x-\frac{3}{4}=$	$7\frac{1}{4}$	$-12\frac{3}{4}$

It is seen from Fig. 41 that this straight line graph cuts the graph of $y=5-3x-x^2$ at the points Q and R whose x co-ordinates are $-1\cdot95$ and $2\cdot95$ respectively. Thus the solution of the equation $4x^2-4x-23=0$ is $x=-1\cdot95$ or $2\cdot95$.

The questions in Exercise 24b, page 357, should now be attempted.

Equations of the Third Degree

175. Example 7. Plot the graph of $y=x^3$ for the range $x=-2$ to 2, and thus solve the equations,

(1) $8x^3-12x-21=0$, (2) $3x^3-9x-4=0$, (3) $x^3-3x+2=0$.

If the point (x, y) lies on the graph $y=x^3$, it is evident that the point $(-x, -y)$ also lies on the graph. As it will be seen that the graph bends rapidly, values are calculated for intervals of $\frac{1}{5}$ in the value of x. The table of values, calculated to 2 places of decimals, is:

$x=0$	$\pm\cdot2$	$\pm\cdot4$	$\pm\cdot6$	$\pm\cdot8$	±1	$\pm1\cdot2$	$\pm1\cdot4$	$\pm1\cdot6$	$\pm1\cdot8$	±2
$y=x^3=0$	$\pm\cdot01$	$\pm\cdot06$	$\pm\cdot22$	$\pm\cdot51$	±1	$\pm1\cdot73$	$\pm2\cdot74$	$\pm4\cdot10$	$\pm5\cdot83$	±8

The graph is shown in Fig. 42.

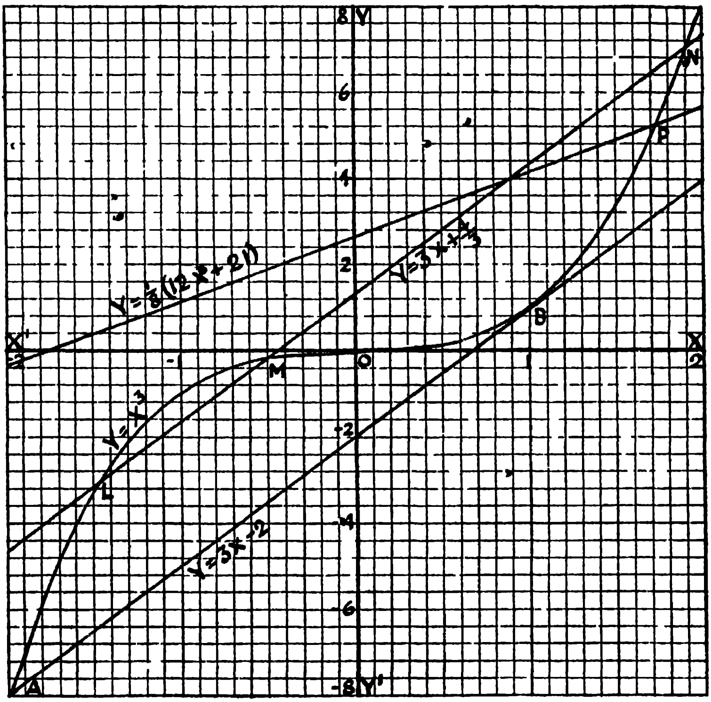

FIG. 42

(1) The equation $8x^3-12x-21=0$ may be put in the form $x^3=\frac{1}{8}(12x+21)$. Hence as we have the graph of $y=x^3$, we need to plot the graph of $y=\frac{1}{8}(12x+21)$, and obtain the x co-ordinates of their points of intersection. The table of values is:

$x=$	-2	2
$y=\frac{1}{8}(12x+21)=$	$-\cdot 38$	$5\cdot 63$

It is seen that the graphs intersect only at the point P, whose x co-ordinate is $1\cdot 74$. Thus the equation $8x^3-12x-21=0$ has only one real root, namely, $x=1\cdot 74$.

(2) The equation $3x^3-9x-4=0$ may be put in the form $x^3=3x+\frac{4}{3}$. Hence we need to plot the graph of $y=3x+\frac{4}{3}$.
The table of values is:

$x=$	-2	2
$y=3x+\frac{4}{3}=$	$-4\cdot 67$	$7\cdot 33$

It is seen that the graphs intersect at the three points L, M, N, whose x co-ordinates are $-1\cdot 44$, $-\cdot 48$, $1\cdot 92$ respectively, which are thus the roots of the equation $3x^3-9x-4=0$.

(3) To solve the equation $x^3-3x+2=0$, we need to draw the graph of $y=3x-2$. The table of values is:

$x=$	-2	2
$y=3x-2=$	-8	4

It is seen that this straight line graph cuts the graph of $y=x^3$ at the point A, whose x co-ordinate is -2, and that it is a tangent to the graph at the point B, whose x co-ordinate is 1.

Thus the equation $x^3-3x+2=0$ has one root $x=-2$, and two equal roots, each of which is $x=1$. (See Example 5 (1).)

The questions in Exercise 24c, page 359, should now be attempted.

Exercise 24a

1. Draw the graph of $y=f(x)$ when $f(x)=x^2-x-6$, for the range $x=-4$ to $x=+4$, and thus solve the equation $x^2-x-6=0$.

2. Draw the graph of $y=f(x)$ when $f(x)=2+x-x^2$, for the range $x=-3$ to $x=+3$. Use the graph to solve the equation

$$2+x-x^2=0.$$

What conclusion do you arrive at from the sign of x^2 in examples (1) and (2) and the way the two graphs lie?

3. Draw the graph of $y=f(x)$ when $f(x)=4x^2+8x-5$, for the range $x=-4$ to $x=+2$.

Solve the equation $4x^2+8x-5=0$ by using this graph.

4. Solve the following equations graphically:

(a) $2x^2-7x+4=0$. (b) $2x^2+9x=0$.
(c) $-4+12x-9x^2=0$. (d) $3x^2-5x-1=0$.
(e) $13-5x^2=0$.

5. Plot the graph of $y=f(x)$ when $f(x)=9+2x-x^2$, for the range $x=-3$ to $x=+5$.

From your graph read the maximum value of the function, and the value of x which makes the function a maximum.

6. Plot the graph of $y=f(x)$ when $f(x)=5x^2+6x-12$, for the range $x=-3$ to $x=+2$.

From the graph find the value of x which makes the function a minimum, and also the minimum value of the function.

For what range of values of x is the value of the function negative?

7. Plot the graphs of the following functions and find the value of x which makes the function a maximum or minimum; also state the maximum or minimum value of the function.

(a) $f(x)=x^2-5x+4$. (b) $f(x)=2x^2-5x+8$.
(c) $f(x)=5-8x-4x^2$. (d) $f(x)=-17+12x-3x^2$.
(e) $f(x)=4x^2+4x+1$.

8. From the graphs (a) to (e) in Question 7 read off the ranges of values of x for which the corresponding functions are positive.

Exercise 24b

1. Plot the graph of $y=f(x)$ when $f(x)=x^2$, for the range $x=-4$ to $x=+4$.

By drawing the appropriate straight lines on this graph solve the equations:

(a) $x^2-2x-3=0$. (b) $2x^2+x-10=0$.
(c) $x^2-4=0$. (d) $3x^2-5x-3=0$.
(e) $4x^2+15x=0$.

2. By drawing the appropriate portions of the graph of (d), Question 1, to a larger scale, find two roots correct to three decimal places.

3. Draw the graph of $y=f(x)$ when $f(x)=x^2$, for the range $x=-3$ to $x=+3$. Use this graph to find, and explain the nature of, the roots of the following equations:

(a) $8x^2+6x-27=0$. (b) $9x^2+24x+16=0$.
(c) $x^2-2x+2=0$. (d) $-4x^2+8x+5=0$.

4. Use the graphs of (a) and (d) of Question 3 to find the range of values of x which make (1) $8x^2+6x-27$ negative, and (2) $5+8x-4x^2$ positive.

5. Draw the graph of $y=x^2$, and on the same axes and to the same scale the graph of the following relations:

(a) $y=x+2$. (b) $y=6-x$.
(c) $y=\frac{3}{2}x$. (d) $y=\frac{15}{4}x-\frac{9}{4}$.

At what points do these lines cut the graph of $y=x^2$?

Write down in each case the equation that has the values of x at the points of intersection as roots.

6. Plot the graph of $y=f(x)$ when $f(x)=x^2-3x$, for the range $x=-3$ to $x=+6$. Use this graph to solve the equations:

(a) $x^2-3x=0$. (b) $x^2-3x+2=0$.
(c) $x^2-3x-10=0$. (d) $4x^2-12x+5=0$.
(e) $4x^2-12x+9=0$.

7. Draw the graph of $y=f(x)$ when $f(x)=12-x-6x^2$, for the range $x=-4$ to $x=+4$.

What is the maximum value of this function?

Use the graph to solve the following equations:

(a) $12-x-6x^2=0$. (b) $5-x-6x^2=0$.
(c) $-1+x+6x^2=0$. (d) $10-3x-18x^2=0$.

8. Plot the graph of $y=x^2+2x-3$ for the range $x=-4$ to $x=2$.

On the same axes and to the same scale plot the graph of $y=x+1$.

(a) What values of x correspond to the points of intersection of these graphs?

(b) Of what equation are these values of x a solution?

9. By drawing the appropriate portions of the two graphs in Question 8 to a larger scale, find the two roots of the equation obtained from these graphs to three places of decimals.

10. Plot the graph of $y=3x^2+x-3$ for the range $x=-3$ to $x=+3$.

On the same axes and to the same scale plot the graph of $y=1-5x$.

What values of x correspond to the points of intersection of these graphs?

Of what equation are the values of x at the points of intersection a solution?

11. By drawing the appropriate portions of the two graphs in Question 10 to a larger scale, find the two roots of the equation obtained from these graphs to three places of decimals.

12. Plot the graph of $y=x^2-3x-4$ for the range $x=-3$ to $x=6$. By drawing an appropriate straight line graph solve the equation $x^2-6x+5=0$.

13. Draw the graph of $y=2x^2+3x-1$ for the range $x=-4$ to $x=+2$. Use this graph to solve the equation $4x^2+16x+7=0$.

14. Plot the graph of $y=2x^2-2x+1$ for the range $x=-2$ to $x=+4$. Use this graph to solve the following equations:

(*a*) $3x^2-x-2=0$. (*b*) $4x^2-20x+25=0$.
(*c*) $2x^2-7x+1=0$.

Exercise 24c

1. Plot the graph of $y=f(x)$ when $f(x)=x^3+2x^2-x-2$, for the range $x=-3$ to $x=+3$.

From the graph solve the equation $x^3+2x^2-x-2=0$.

2. Plot the graphs of the following functions:

(*a*) $f(x)=2x^3+x^2-x$. (*b*) $f(x)=(2x-3)^2(x+2)$.
(*c*) $f(x)=5x^2-2x^3$. (*d*) $f(x)=x^3-4x^2+5x$.
(*e*) $f(x)=(2x-1)(x^2+1)$.

Write down the values of x that make $f(x)=0$ in each case.

3. Draw the graph of $y=x^3$ for the range $x=+5$ to $x=-5$.

With the same scale and on the same axes draw the graph of $y=13x-12$.

Verify by substitution that the values of x at the points of intersection of these graphs satisfy the equation

$$x^3-13x+12=0.$$

4. Plot the graph of $y=x^3$ for the range $x=+4$ to $x=-4$, and by drawing the appropriate straight line on your graph, solve the equations:

(*a*) $x^3-7x-6=0$. (*b*) $x^3+x+2=0$.
(*c*) $x^3-3x-2=0$.

5. Plot the graph of $y=\frac{8x+3}{2x-1}$ for the range $x=-4$ to $x=0$.

What is the value of x where the graph cuts the x-axis?

6. Use the graph of Question 5 together with the graph of the appropriate straight line to solve the equation

$$x+\frac{8x+3}{2x-1}=0.$$

7. Plot the graph of $y=\frac{4}{x}$ for the range $x=+4$ to $x=-4$.

On the same axes and to the same scale draw the graph of $y=3x-1$.

Show that the values of x at the points of intersection of your graph satisfy the equation $3x^2-x-4=0$.

8. Use the graphs of the previous question to solve the simultaneous equations:

$$xy=4,$$
$$3x-y=1.$$

9. Draw the graph of $x^2+y^2=9$ for the range $x=+3$ to $x=-3$. Use your graph to solve the simultaneous equations:

$$x^2+y^2=9,$$
$$y=x+1.$$

giving your answer correct to two places of decimals.

(Before making a table of values put the equation $x^2+y^2=9$ in the form $y=\pm\sqrt{9-x^2}$.)

10. Solve the following pairs of simultaneous equations graphically:

(a) $xy=6,\; x-y=1.$ (b) $x^2+y^2=50,\; xy=7.$ (c) $x^2+y^2=8,\; y-x=-4.$